
Xian-Ming Zhang, Senior Member, IEEE, Qing-Long Han, Fellow, IEEE, and Xiaohua Ge, Member, IEEE

Abstract—One of challenging issues on stability analysis of time-delay systems is how to obtain a stability criterion from a matrix-valued polynomial on a time-varying delay. The first contribution of this paper is to establish a necessary and sufficient condition on a matrix-valued polynomial inequality over a certain closed interval. The degree of such a matrix-valued polynomial can be an arbitrary finite positive integer. The second contribution of this paper is to introduce a novel Lyapunov-Krasovskii functional, which includes a cubic polynomial on a time-varying delay, in stability analysis of time-delay systems. Based on the novel Lyapunov-Krasovskii functional and the necessary and sufficient condition on matrix-valued polynomial inequalities, two stability criteria are derived for two cases of the time-varying delay. A well-illustrated numerical example is given to show that the proposed stability criteria are of less conservativeness than some existing ones.

Index Terms—Bessel-Legendre inequality, matrix-valued polynomial inequalities, stability, time-varying delay, time-delay systems.

I. INTRODUCTION

TIME-DELAY systems have received considerable attention in the field of control during the past two decades. On the one hand, time-delay systems have found more and more applications in industrial control. For example, networked control systems [1] including active control systems for unmanned marine vehicles and offshore platforms in network environments [2], [3] can be modeled as time-delay systems. The problem of coordination and formation control of multi-agent systems can be solved by employing time-delay system theory [4]. On the other hand, although it is well known that time-delays usually play the negative effects on a control system, their potential positive effects are often disclosed. It is proven that for networked harmonic oscillators, synchronization cannot be reached using current position data, but can be achieved using delayed position data [5]. For offshore platforms, by intentionally introducing a small time-delay into the feedback channel, oscillation amplitudes and control forces can be reduced significantly [3]. Therefore, time-delay systems are still an important topic to research both in theory and in practice.

Delay-dependent stability of time-delay systems has been studied for a long time, see, e.g. [6]–[11]. Its objective is to derive a stability condition such that the allowable delay upper bound is as large as possible. To achieve this goal, a number of notable methods (techniques) have been proposed, such as a free-weighting matrix approach, an integral inequality approach, a quadratic convex approach, and a reciprocally convex combination inequality and a Wirtinger-based inequality [12], see the survey paper [13]. Since 2013, boosted by the Wirtinger-based inequality, much progress has been made on delay-dependent stability analysis of time-delay systems. One can obtain some less conservative stability criteria using a Bessel-Legendre inequality, which is an extension of the Wirtinger-based inequality. However, when a Bessel-Legendre inequality is used, the time-derivative of some certain Lyapunov-Krasovskii functional $\mathcal{V}(t)$ may be estimated as a polynomial with respect to the time-varying delay $d(t) \in [0, \bar{h}]$, that is

$$\dot{\mathcal{V}}(t) \leq \xi^T(t) f(d(t)) \xi(t), \quad f(d(t)) = \sum_{i=0}^{m} \bar{d}^i d(t)^i, \quad \bar{d} \in \mathbb{R}^m,$$

(1)

where $\bar{d}_i (i = 0, 1, \ldots, m)$ with $m \geq 2$ are symmetric real matrices irrespective of $d(t)$; $\xi(t)$ is a state-related vector, and $\bar{h}$ is a positive constant. Then a hard nut to crack is how to derive a stability criterion from the matrix inequality $f(d(t)) < 0$ for $d(t) \in [0, \bar{h}]$. Although some sufficient conditions on $f(d(t)) < 0$ with $m = 2$ for $d(t) \in [0, \bar{h}]$ are presented in [14], [15], a necessary and sufficient condition on such a matrix inequality has not been reported yet, which motivates the current study.

In this paper, we first establish a necessary and sufficient condition on $f(d(t)) < 0$ (or $f(d(t)) > 0$) for $d(t) \in [0, \bar{h}]$. Then, the obtained necessary and sufficient condition is applied to stability analysis of time-delay systems. If the time-varying delay $d(t)$ is differentiable, and its derivative function is bounded from below and above, a novel Lyapunov-Krasovskii functional with a cubic matrix-valued polynomial like $\dot{d}^2(d(t)) P_3 + d^2(d(t)) P_2 + d(d(t)) P_1 + P_0$ is introduced. If the time-varying delay $d(t)$ is just continuous while not differentiable, a novel Lyapunov-Krasovskii functional is also introduced. A common feature of these novel Lyapunov-Krasovskii functionals is that their time-derivatives are estimated as $\xi^T(t) \tilde{f}(d(t)) \xi(t)$, where $\tilde{f}(d(t))$ is a quartic matrix-valued polynomial as $\sum_{j=0}^{4} d^j(t) \Phi_j$ with $\Phi_j (j = 0, 1, \ldots, 4)$ being symmetric real matrices irrespective of $d(t)$. The obtained
necessary and sufficient condition \( f(d(t)) < 0 \) for \( d(t) \in [0, \bar{h}] \) is utilized to deliver some less conservative stability criteria, which is demonstrated through a well-studied numerical example.

**Notations:** The notations throughout this paper are standard. \( \text{diag}\{\cdots\} \) and \( \text{col}\{\cdots\} \) denote a block-diagonal matrix and a block-column matrix (vector), respectively. The set \( \mathbb{S}^n(\mathbb{S}^n_+^m) \) represents the set of symmetric (positive definite) matrices of \( \mathbb{R}^{nxn} \). \( \text{He}[X] = X + XT \).

**II. NECESSARY AND SUFFICIENT CONDITION ON MATRIX-VALUED POLYNOMIAL INEQUALITIES**

Consider the following matrix-valued polynomial described by

\[
F_m(s) = s^{2m} \Phi_{2m} + s^{2m-1} \Phi_{2m-1} + \cdots + \Phi_0
\]

where \( m \geq 1 \) is an integer, and \( s \in [0, \bar{h}] \) with \( \bar{h} \) being a constant; and \( \Phi_j \in \mathbb{S}^{nxn} \) \( (j = 0, 1, 2, \ldots, 2m) \). If \( \Phi_{2m} = 0 \),

\[
F_m(s) = s^{2m-1} \Phi_{2m-1} + s^{2m-2} \Phi_{2m-2} + \cdots + \Phi_0
\]

which is an odd matrix-valued polynomial on \( s \) if \( \Phi_{2m-1} \neq 0 \). Thus, (2) represents both even and odd matrix-valued polynomials.

For \( m = 1, F_1(s) \) with \( \Phi_2 = 0 \) is convex on \( s \in [0, \bar{h}] \). However, \( F_1(s) \) with \( \Phi_2 \neq 0 \) is not necessarily convex on \( s \in [0, \bar{h}] \). Thus, an emerging topic in recent years is to seek conditions such that \( F_1(s) \) with \( \Phi_2 \neq 0 \) is strictly less than zero for \( \forall s \in [0, \bar{h}] \), see, e.g., [14], [15]. In the following, we present a necessary and sufficient condition on \( F_m(s) < 0 \) for \( \forall s \in [0, \bar{h}] \). To begin with, we introduce a key lemma as follows.

**Lemma 1:** For given matrices \( \Omega \in \mathbb{S}^p, H_1, H_2 \in \mathbb{R}^{kp} \) with \( p > k \), the following statements are equivalent:

1. The inequality \( \zeta^T \Omega \zeta < 0 \) holds for all nonzero vectors \( \zeta \in \mathbb{R}^p \) that satisfy \( (H_2 - \delta H_1) \zeta = 0 \) for some real scalar \( \delta \) such that \( |\delta| \leq 1 \).
2. There exist a matrix \( D \in \mathbb{S}_+^k \), and a skew-symmetric matrix \( G \in \mathbb{R}^{k \times k} \) such that

\[
\begin{bmatrix}
H_1^T & D \\
D^T & -D
\end{bmatrix}
\begin{bmatrix}
H_1 \\
H_2
\end{bmatrix} + \Omega < 0.
\]

3. There exists \( X \in \mathbb{R}^{kp \times k} \) such that

\[
X(H_2 + H_1) + (H_2 + H_1)^T X^T + \Omega < 0
\]

\[
X(H_2 - H_1) + (H_2 - H_1)^T X^T + \Omega < 0
\]

4. There exist \( \varepsilon_i \in \mathbb{R} \) \( (i = 1, 2) \) such that

\[
\Omega - \varepsilon_1 (H_2 + H_1)^T (H_2 + H_1) < 0
\]

\[
\Omega - \varepsilon_2 (H_2 - H_1)^T (H_2 - H_1) < 0
\]

**Proof:** The equivalence between 1), 2) and 3) can be found in [16]. The equivalence between 3) and 4) is derived from the Finsler Lemma [17].

Let \( \zeta = \text{col}\{I, sI, s^2I, \cdots, s^mI\} \zeta_0 \) with \( \zeta_0 \in \mathbb{R}^q \) and \( \zeta_0 \neq 0 \). Then

\[
f_m(s) = \zeta_0^T F_m(s) \zeta_0 = \zeta^T \Omega_m \zeta,
\]

where

\[
\Omega_m = \begin{bmatrix}
\Phi_0 & \frac{1}{2} \Phi_1 & \cdots & 0 & 0 \\
\frac{1}{2} \Phi_1 & \Phi_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \Phi_{2m} \\
0 & 0 & \cdots & \frac{1}{2} \Phi_{2m-1} & \Phi_{2m}
\end{bmatrix}
\]

By applying Lemma 1, we have the following result.

**Theorem 1:** For the matrix-valued polynomial \( F_m(s) \) in (2), then

i) \( F_m(s) < 0 \) for \( \forall s \in [0, \bar{h}] \) if and only if there exist an \( X \in \mathbb{S}_+^m \) and a skew-symmetric matrix \( S \in \mathbb{R}^{mp \times mp} \) such that

\[
\Omega_m + [H_1 \quad H_2] \begin{bmatrix}
X & S \\
S^T & -X
\end{bmatrix} [H_1 \\
H_2] < 0;
\]

ii) \( F_m(s) > 0 \) for \( \forall s \in [0, \bar{h}] \) if and only if there exist an \( X \in \mathbb{S}_+^m \) and a skew-symmetric matrix \( S \in \mathbb{R}^{mp \times mp} \) such that

\[
\Omega_m - [H_1 \quad H_2] \begin{bmatrix}
X & S \\
S^T & -X
\end{bmatrix} [H_1 \\
H_2] > 0,
\]

where \( \Omega_m \) is given in (9); and

\[
H_1 = \begin{bmatrix}
\bar{h}I & 0 & \cdots \\
\bar{h}I & 0 & \cdots \\
\vdots & \vdots & \ddots \\
\bar{h}I & 0 & \cdots
\end{bmatrix}_{mp \times (m+1)q}
\]

\[
H_2 = \begin{bmatrix}
\bar{h}I & -2I & \cdots \\
\bar{h}I & -2I & \cdots \\
\vdots & \vdots & \ddots \\
\bar{h}I & -2I & \cdots
\end{bmatrix}_{mp \times (m+1)q}
\]

**Proof:** i) Note that

\[
(H_2 - \delta H_1) \zeta = \begin{bmatrix}
[\bar{h}(1-\delta) - 2s]I \\
[\bar{h}(1-\delta) - 2s]I \\
\vdots \\
[\bar{h}(1-\delta) - 2s]I
\end{bmatrix} \zeta_0.
\]

Then \( (H_2 - \delta H_1) \zeta = 0 \) for some real scalar \( \delta \) such that \( |\delta| \leq 1 \) if and only if \( s \in [0, \bar{h}] \). In fact

\[
(H_2 - \delta H_1) \zeta = 0 \iff \bar{h} \delta = \bar{h} - 2s.
\]

Then it is not difficult to verify that

\[
|\bar{h} \delta| = |\bar{h} - 2s| \leq \bar{h} \iff 0 \leq s \leq \bar{h}.
\]

Applying Lemma 1, \( f_m(s) < 0 \) for \( \forall s \in [0, \bar{h}] \) if and only if there exist an \( X \in \mathbb{S}_+^m \) and a skew-symmetric matrix \( S \in \mathbb{R}^{mp \times mp} \) such that (10) is satisfied, which completes the proof of i).

ii) Set \( \tilde{F}_m(s) = -F_m(s) \). Then the proof is straightforward from the proof of i).
\[
\begin{bmatrix}
\Phi_0 & \frac{1}{2} \Phi_1 \\
\frac{1}{2} \Phi_1 & \Phi_2 \\
\end{bmatrix} + \begin{bmatrix} \tilde{h}I & 0 \\ \tilde{h}I & -2I \end{bmatrix} \begin{bmatrix} X & S \\ S^T & -X \end{bmatrix} \begin{bmatrix} \tilde{h}I & 0 \\ \tilde{h}I & -2I \end{bmatrix} < 0.
\]

which is equivalent to that in [18].

In the next section, we are to establish some novel stability criteria for time-delay systems by using Theorem 1. To end this section, we introduce a canonical Bessel-Legendre inequality as follows [19], [20].

**Lemma 2:** For an integer \( N \geq 0 \), two scalars \( a \) and \( b \) with \( b > a \), an \( n \times n \) real matrix \( R > 0 \), and a differentiable function \( x : [a,b] \rightarrow \mathbb{R}^n \) such that the integrations below are well defined, then

\[
-(b-a) \int_a^b \dot{x}^T(s) R \dot{x}(s) ds \leq -\sigma \Lambda_N^T \Theta_N \Phi_N \Theta_N \Lambda_N \sigma_N
\]

where

\[
\begin{align*}
\Phi_N &= \begin{bmatrix}
I & 0 & \cdots & 0 \\
I & (-1)^{1} \left( \frac{1}{1} \right) I & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
I & (-1)^{N-1} \left( \frac{N}{N} \right) I & \cdots & (-1)^{N} \left( \frac{2N}{N} \right) I
\end{bmatrix} \\
\Theta_N &= \begin{bmatrix}
I & -I & \cdots & 0 \\
0 & I & \cdots & 0 \\
0 & -I & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & -I & \cdots & N I
\end{bmatrix} \\
\Lambda_N &= \begin{bmatrix}
I & -I & 0 & \cdots & 0 \\
0 & I & 0 & \cdots & 0 \\
0 & -I & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & -I & \cdots & \cdots & 0 \\
\end{bmatrix}
\end{align*}
\]

\[
\sigma_N = \text{col}\{x(b), x(a), \gamma_1(a,b), \ldots, \gamma_N(a,b)\}
\]

\[
\gamma_k(a,b) = \int_a^b \frac{(b-s)^{k-1}}{(b-a)^k} x(s) ds, \quad (k = 1, 2, \ldots, N).
\]

### III. APPLICATION TO STABILITY ANALYSIS OF TIME-DELAY SYSTEMS

Consider the following time-delay system described by

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B x(t-d(t)) \\
\phi(\theta) &= \phi_0, \quad \theta \in [-\tilde{h}, 0]
\end{align*}
\]

where \( x(t) \in \mathbb{R}^n \) is the system state and \( \phi_0 \) is the initial condition; \( A, B \in \mathbb{R}^{n \times n} \). Suppose that the time delay \( d(t) \) satisfies one of two cases as

**Case 1:** \( d(t) \) is a differentiable function satisfying

\[
0 \leq d(t) \leq \tilde{h}, \quad \mu_1 \leq d(t) \leq \mu_2
\]

**Case 2:** \( d(t) \) is a continuous function satisfying

\[
0 \leq d(t) \leq \tilde{h}
\]

where \( \tilde{h}, \mu_1 \) and \( \mu_2 \) are constants with \( \mu_1 < 0 \) and \( \mu_2 > 0 \).

**A. Stability Criteria for the System (14) in Case 1**

To begin with, we denote

\[
\begin{align*}
\gamma_1(t) &= \text{col}\{v_{11}(t), v_{12}(t), v_{13}(t), v_{14}(t)\} \\
\gamma_2(t) &= \text{col}\{v_{21}(t), v_{22}(t), v_{23}(t), v_{24}(t)\}
\end{align*}
\]

where

\[
\begin{align*}
\gamma_1(t) &= \int_{t-d(t)}^{t} (t-s)^{\mu_1-1} x(s) ds \\
\gamma_2(t) &= \int_{t-h}^{t} \frac{(t-d(t)-s)^{\mu_2-1} x(s) ds}{(h-d(t))^{\mu_2}}
\end{align*}
\]

Construct the following Lyapunov-Krasovskii functional candidate as

\[
V(t,x) = V_1(t,x) + V_2(t,x) + V_3(t,x)
\]

where

\[
\begin{align*}
V_1(t,x) &= \psi^T(t) P(d(t)) \psi_1(t) \\
V_2(t,x) &= \int_{t-d(t)}^{t} \psi_2^T(s,x(t-d(t))) Q_1 \psi_2(s,x(t)) ds \\
V_3(t,x) &= \tilde{h} \int_{t-h}^{t} \psi_3^T(h-t+s) x^T(s) R \tilde{x}(x(s)) ds \\
& \quad + \tilde{h} \int_{t-h}^{t} \frac{(h-t+s)}{(h-d(t))^{\mu_2}} x^T(s) R \tilde{x}(x(s)) ds
\end{align*}
\]

where \( Q_1 > 0, Q_2 > 0, R_1 > 0, R_2 > 0 \) and

\[
\begin{align*}
P(d(t)) &= d^3 P_3 + d^2 P_2 + d P_1 + P_0 \\
\psi_1(t) &= \text{col}\{\phi(t), d(t) v_1(t), (\tilde{h}-d(t)) v_2(t)\} \\
\psi_0(t) &= \text{col}\{x(t), x(t-d(t)), (t-h)\} \\
\psi_2(s,t) &= \text{col}\{\dot{x}(s), x(s), (\tilde{h}-d(t)) v_2(t)\} \\
& \quad \int_{t-d(t)}^{t} (s-\theta) x(\theta)d\theta, \int_{s}^{t} x(\theta)d\theta, \int_{t-d(t)}^{t} x(s)d\theta \\
& \quad \int_{t-d(t)}^{t} (s-\theta) x(\theta)d\theta, \int_{s}^{t} x(\theta)d\theta, \int_{t-h}^{t} x(\theta)d\theta, \int_{t-h}^{t} x(s)d\theta
\end{align*}
\]

with \( P_i \in \mathbb{S}^{11n} \).

**Remark 2:** The Lyapunov-Krasovskii functional \( V(t,x) \) is different from some existing ones published in the literature. On the one hand, a cubic matrix-valued polynomial \( P(d(t)) \) in (19) is introduced in \( V_1(t,x), \) leading to the fact that \( V_1(t,x) \) is a cubic polynomial on \( d(t) \). The positive finiteness of \( P(d(t)) \) for \( \forall d(t) \in [0,\tilde{h}] \) does not need all \( P_i (i = 0, 1, 2, 3) \) to be positive definite. By Theorem 1 (ii), \( P(d(t)) > 0 \) for \( \forall d(t) \in [0,\tilde{h}] \) if only if there exist an \( X > 0 \) and a skew-symmetric real matrix \( S \) such that

\[
\begin{bmatrix}
P_0 & \frac{1}{2} P_1 \\
\frac{1}{2} P_1 & P_2 \\
\frac{1}{2} P_2 & P_3 
\end{bmatrix}
\begin{bmatrix} X & S \\ S^T & -X \end{bmatrix}
\begin{bmatrix} X \\ S \end{bmatrix} > 0
\]

which, by Lemma 1, is equivalent to that there exist \( \epsilon_i \in \mathbb{R} (i = 1, 2) \) such that

\[
\begin{bmatrix}
P_0 & \frac{1}{2} P_1 \\
\frac{1}{2} P_1 & P_2 \\
\frac{1}{2} P_2 & P_3 
\end{bmatrix}
\begin{bmatrix} X & S \\ S^T & -X \end{bmatrix}
\begin{bmatrix} X \\ S \end{bmatrix} + \epsilon_1 (H_2 + \tilde{H}_1)^T (H_2 + \tilde{H}_1) > 0
\]

and

\[
\begin{bmatrix}
P_0 & \frac{1}{2} P_1 \\
\frac{1}{2} P_1 & P_2 \\
\frac{1}{2} P_2 & P_3 
\end{bmatrix}
\begin{bmatrix} X & S \\ S^T & -X \end{bmatrix}
\begin{bmatrix} X \\ S \end{bmatrix} + \epsilon_2 (H_2 - \tilde{H}_1)^T (H_2 - \tilde{H}_1) > 0
\]
\[ \rho_1 = \begin{bmatrix} h & 0 & 0 \\ 0 & h & 0 \end{bmatrix}, \quad \rho_2 = \begin{bmatrix} h & -2I & 0 \\ 0 & h & -2I \end{bmatrix}. \]

On the other hand, four vectors \[ \int_{-d(t)}^{0} (s \theta - \theta) x(t) dt, \int_{0}^{s} (\theta - s) x(t) dt \text{ and } \int_{-d(t)}^{0} (s \theta - \theta) x(t) dt \text{ and } \int_{0}^{s} (\theta - s) x(t) dt \] are included in \( V_2(\alpha, \xi) \), which brings more information on past system states into the derivative of the Lyapunov-Krasovskii functional.

Based on the Lyapunov-Krasovskii functional \( V(t, \xi) \), we now state and establish the following result.

**Proposition 1:** For given constants \( \mu_1, \mu_2 \), and \( \dot{h} \), the system described by (14) and (15) is asymptotically stable if there exist \( Q_1 > 0, R_1 > 0 \), symmetric real matrices \( P_0, P_1, P_2, P_3, Z_0 \), real matrices \( Y_i \) with appropriate dimensions and scalars \( e_i, e_{i1}, e_{i2} \) (i = 1, 2) such that (20), (21) and

\[
\begin{align*}
\begin{bmatrix} R_1 - Z_1 & Y_1 \\ Y_1^T & R_2 \end{bmatrix} & \geq 0, \\
\begin{bmatrix} R_1 & Y_2 \\ Y_2^T & R_2 - Z_2 \end{bmatrix} & \geq 0,
\end{align*}
\]

(22)

\[
\Phi(\mu_1) - e_1((H_2 + H_1)^T(H_2 - H_1)) < 0
\]

(23)

where \( R_i = \text{diag}[R_i, 3R_i, 5R_i, 7R_i, 9R_i] \) (i = 1, 2), and

\[
\begin{align*}
H_1 &= \begin{bmatrix} h & 0 & 0 \\ 0 & h & 0 \end{bmatrix}, \\
H_2 &= \begin{bmatrix} h & -2I & 0 \\ 0 & h & -2I \end{bmatrix},
\end{align*}
\]

(22)

\[
\Phi(\mu_1) - e_1((H_2 - H_1) - H_1)^T(H_2 - H_1) < 0
\]

(23)

where \( R_i = \text{diag}[R_i, 3R_i, 5R_i, 7R_i, 9R_i] \) (i = 1, 2), and

\[
\begin{align*}
\begin{bmatrix} R_1 & Z_1 & Y_1 \\ Y_1^T & R_2 & Z_2 \end{bmatrix} & \geq 0, \\
\begin{bmatrix} R_1 & Y_2 \\ Y_2^T & R_2 \end{bmatrix} & \geq 0,
\end{align*}
\]

(22)

\[
\Phi(\mu_1) - e_1((H_2 + H_1)^T(H_2 - H_1)) < 0
\]

(23)

where \( R_i = \text{diag}[R_i, 3R_i, 5R_i, 7R_i, 9R_i] \) (i = 1, 2), and

\[
\begin{align*}
\begin{bmatrix} R_1 - Z_1 & Y_1 \\ Y_1^T & R_2 \end{bmatrix} & \geq 0, \\
\begin{bmatrix} R_1 & Y_2 \\ Y_2^T & R_2 - Z_2 \end{bmatrix} & \geq 0,
\end{align*}
\]

(22)

\[
\Phi(\mu_1) - e_1((H_2 + H_1)^T(H_2 - H_1)) < 0
\]

(23)

where \( R_i = \text{diag}[R_i, 3R_i, 5R_i, 7R_i, 9R_i] \) (i = 1, 2), and

\[
\begin{align*}
\begin{bmatrix} R_1 & Z_1 & Y_1 \\ Y_1^T & R_2 & Z_2 \end{bmatrix} & \geq 0, \\
\begin{bmatrix} R_1 & Y_2 \\ Y_2^T & R_2 \end{bmatrix} & \geq 0,
\end{align*}
\]

(22)

\[
\Phi(\mu_1) - e_1((H_2 + H_1)^T(H_2 - H_1)) < 0
\]

(23)
$D_{60} = \text{col} \{ \bar{h}_7, \bar{h}_2, \bar{h}_1, \bar{h}_3, \bar{h}_5, \bar{h}_4 \}$

$D_{61} = \text{col} \{ \bar{h}_7, 2\bar{h}_1, \bar{h}_3, 3\bar{h}_2, 4\bar{h}_3, 4\bar{h}_4 \}$

$D_{62} = \text{col} \{ 0, \bar{h}_1, \bar{h}_2, 3\bar{h}_3, 3\bar{h}_2, 6\bar{h}_4, 6\bar{h}_2 \}$

$D_{63} = \text{col} \{ e_3, 2\bar{h}_3, 4\bar{h}_5, 4\bar{h}_4 \}$

with $\rho_0 = e_71 - e_3$ and

$\sigma_1 = e_61 - e_62$, $\sigma_2 = \frac{1}{2}(e_61 - 2e_62 + e_66)$

$\sigma_3 = \frac{1}{2}(e_61 - 2e_62)$, $\sigma_4 = \frac{1}{6}(e_61 - 3e_63 + 3e_62 - e_66)$

$\sigma_5 = \frac{1}{6}(e_61 - 3e_63)$, $\sigma_6 = \frac{1}{6}(e_61 - 3e_63)$

$\sigma_7 = e_61 - e_2$, $\sigma_8 = \frac{1}{6}(e_61 - 3e_62 + 2e_66)$

$\rho_1 = e_71 - e_72$, $\rho_2 = \frac{1}{2}(e_71 - 2e_72 + e_71)$

$\rho_3 = \frac{1}{2}(e_71 - 2e_72)$, $\rho_4 = \frac{1}{6}(e_71 - 3e_73 + e_72 - e_71)$

$\rho_5 = \frac{1}{6}(e_71 - 3e_73)$, $\rho_6 = \frac{1}{6}(e_71 - 3e_73)$

$\rho_7 = e_71 - e_2$, $\rho_8 = \frac{1}{6}(e_71 - 3e_72 + e_71)$

Proof: First, the conditions (20) and (21) ensure that the real matrix $P(d(t))$ is positive definite for $\forall d(t) \in [0, \bar{h}]$. Thus, $V(t, x(t))$ constructed in (18) is a Lyapunov-Krasovskii functional candidate. Then, we take the derivative of $V(t, x(t))$ along with the trajectory of the system (14) to obtain

$$V(t, x(t)) = V_1(t, x(t)) + V_2(t, x(t)) + V_3(t, x(t))$$

where

$$V_1(t, x(t)) = 2\psi_1^T(t)P(d(t))\dot{\psi}_1(t) + \psi_1^T(t)P(d(t))\dot{\psi}_1(t)$$

$$V_2(t, x(t)) = \psi_2^T(t)Q_1\dot{\psi}_2(t) - \psi_3^T(t)Q_2\dot{\psi}_3(t - \bar{h}, t)$$

$$+ (1 - \dot{d}(t))\psi_2^T(t - d(t), t)Q_1\dot{\psi}_2(t - d(t), t)$$

$$+ (1 - \dot{d}(t))\psi_3^T(t - d(t), t)Q_2\dot{\psi}_3(t - d(t), t)$$

$$+ \int_{t-d(t)}^{t} 2\psi_2^T(s)Q_1\frac{\partial}{\partial t}\psi_2(s, t) ds$$

$$+ \int_{t-d(t)}^{t} 2\psi_3^T(s)Q_2\frac{\partial}{\partial t}\psi_3(s, t) ds$$

$$V_3(t, x(t)) = \bar{h}^T(t)(R_1 x(t) + \bar{h}(t - d(t))(\dot{h} - d(t)))$$

$$\times \chi^T(t - d(t))(R_2 - R_1)\dot{x}(t - d(t))$$

$$+ I_1(t) + I_2(t)$$

where $I_1(t) = -\bar{h} \int_{t-d(t)}^{t} \dot{x}^T(s)R_1\dot{x}(s) ds$ and $I_2(t) = \bar{h} \int_{t-d(t)}^{t} \dot{x}^T(s)R_2\dot{x}(s) ds$. Denote

$$\xi(t) = \text{col} \{ x(t), x(t - d(t)), x(t - \bar{h}), \dot{x}(t - d(t)),$$

$$\dot{x}(t - \bar{h}), v_1(t), v_2(t) \}$$

where $v_1(t)$ and $v_2(t)$ are defined in (17). Note that

$$\frac{d}{dt}[\dot{d}(t)]v_1(t) = I_1(d(t))\xi(t)$$

$$\frac{d}{dt}[(\bar{h} - d(t))v_2(t)] = I_2(d(t))\xi(t)$$

where $I_1(\cdot)$ and $I_2(\cdot)$ are defined in Proposition 1. Hereafter, the notations used are also defined in Proposition 1 without declaration. Then

$$\psi_1(t) = C_1\xi(t), \ \dot{\psi}_1(t) = C_2\xi(t)$$

where $C_1 = C_{11} + d(t)C_{12}$. Hence, we have that

$$\dot{V}_1(t, x(t)) = \xi^T(t)\dot{\Psi}_1(d(t), d(t))\xi(t)$$

where

$$\Psi_1(d(t), d(t)) = \dot{d}(t)C_1^T[3\dot{d}(t)P_3 + 2d(t)P_2 + P_1]C_1$$

$$+ \text{He}[C_1^T P(d(t))C_2]$$

Some algebraic manipulations follow that

$$\psi_2(t, t) = C_3\xi(t), \ \dot{\psi}_3(t - \bar{h}, t) = C_4\xi(t)$$

$$\psi_2(t - d(t), t) = C_5\xi(t), \ \psi_3(t - \bar{h}, t, t) = C_6\xi(t)$$

where

$$C_1 = C_{11} + d(t)C_{12} + \dot{d}(t)C_{13}, \ j = 3, 4, 5, 6$$

Since

$$\frac{\partial}{\partial t}\psi_2(s, t) = [N_{11} + g_1(s)N_{12} + (t - s)N_{13}]\xi(t)$$

$$\frac{\partial}{\partial t}\psi_3(s, t) = [N_{21} + g_1(s)N_{22} + g_2(s)N_{23}]\xi(t)$$

where $g_1(s) = t - d(t) - s$ and $g_2(s) = t - \bar{h} - s$. Then

$$\int_{t-d(t)}^{t} 2\psi_2^T(s, t)Q_1\frac{\partial}{\partial t}\psi_2(s, t) ds$$

$$+ 2\psi_2^T(t)N_{11}Q_1\int_{t-d(t)}^{t} \psi_2(s, t) ds$$

$$+ 2\psi_2^T(t)N_{12}Q_1\int_{t-d(t)}^{t} g_1(s)\psi_2(s, t) ds$$

and

$$\int_{t-d(t)}^{t} 2\psi_3^T(s, t)Q_2\frac{\partial}{\partial t}\psi_3(s, t) ds$$

$$+ 2\psi_3^T(t)N_{21}Q_2\int_{t-d(t)}^{t} \psi_3(s, t) ds$$

$$+ 2\psi_3^T(t)N_{22}Q_2\int_{t-d(t)}^{t} g_1(s)\psi_3(s, t) ds$$

Note that

$$\int_{t-d(t)}^{t} g_1(s)\psi_2(s, t) ds = D_2(d(t))\xi(t) = \sum_{i=0}^{3} d_i(t)D_{1i}\xi(t)$$

$$\int_{t-d(t)}^{t} g_2(s)\psi_3(s, t) ds = D_3(d(t))\xi(t) = \sum_{i=0}^{4} d_i(t)D_{2i}\xi(t)$$

$$\int_{t-d(t)}^{t} \psi_2(s, t) ds = D_4(d(t))\xi(t) = \sum_{i=0}^{3} d_i(t)D_{3i}\xi(t)$$

$$\int_{t-d(t)}^{t} \psi_3(s, t) ds = D_5(d(t))\xi(t) = \sum_{i=0}^{4} d_i(t)D_{4i}\xi(t)$$

$$\int_{t-d(t)}^{t} g_2(s)\psi_3(s, t) ds = D_6(d(t))\xi(t) = \sum_{i=0}^{4} d_i(t)D_{5i}\xi(t)$$

which lead to
\[ \dot{V}_2(t, x_t) + \dot{V}_3(t, x_t) = \xi^T(t) \Psi_2(d(t), d(t)) \xi(t) + I_1(t) + I_2(t) \] (27)

where

\[
\Psi_2(d(t), \dot{d}(t)) = (1 - d(t))(C_1^T P_2 C_6 - C_5^T Q_1 C_8) + \text{He}\left[ \sum_{j=1}^{3} [N_{1j}^T Q_1 D_j(d(t)) + N_{2j}^T Q_2 D_{3+j}(d(t))] \right] + C_3^T Q_3 C_3 - C_4^T Q_2 C_4 + \dot{h}^2 C_0 \dot{R}_1 C_0 + \dot{h}(1 - d(t))(h - d(t))e_0^T R_2 - R_1 e_4,
\]

For the integral term in (27), applying Lemma 2, one has

\[
I_1(t) \leq \frac{1}{\alpha} \xi^T(t) C_7^T R_1 C_7 \xi(t)
\]

\[
I_2(t) \leq \frac{1}{1 - \alpha} \xi^T(t) C_8^T R_2 C_8 \xi(t)
\]

where \( \alpha = d(t)/\bar{h} \) and \( C_7 = \Theta_4 \Lambda_4 \text{col}[e_1, e_2, e_6], \ C_8 = \Theta_4 \Lambda_4 \text{col}[e_2, e_3, e_7] \).

Use the improved reciprocally convex inequality \([21]\) to get

\[
\dot{V}_1(t, x_t) \leq \xi^T(t) \Psi_3(d(t)) \xi(t) \] (28)

where

\[
\Psi_3(d(t)) = \Psi_3 + d(t) \Psi_3^2 \text{ with} \]

\[
\Psi_3 = -C_7^T (R_1 + Z_1) C_7 - C_8^T R_2 C_8 - \text{He}\left[ C_7^T Y_2 C_8 \right]
\]

\[
\Psi_3^2 = \frac{1}{\bar{h}} \left[ C_7^T Z_1 C_7 - C_8^T Z_2 C_8 - \text{He}\left[ C_7^T (Y_1 - Y_2) C_8 \right] \right]
\]

Substituting (28) into (27) and into (25) yields

\[
\dot{V}(t, x_t) \leq \xi^T(t) \Psi(d(t), \dot{d}(t)) \xi(t)
\] (29)

where

\[
\Psi(d(t), \dot{d}(t)) = \Psi_1(d(t), d(t)) + \Psi_2(d(t), d(t)) + \Psi_3(d(t)) + \sum_{i=0}^{4} d^i(t) P_i
\] (30)

Since \( \Psi(d(t), \dot{d}(t)) \) is linear on \( \dot{d}(t) \in [\mu_1, \mu_2] \), \( \Psi(d(t), \dot{d}(t)) < 0 \) for \( \dot{d}(t) \in [\mu_1, \mu_2] \) if and only if \( \Psi(d(t), \mu_1) < 0 \) and \( \Psi(d(t), \mu_2) < 0 \). By Theorem 1, \( \Psi(d(t), \mu_1) < 0 \) and \( \Psi(d(t), \mu_2) < 0 \) are equivalent to that there exist \( L_1 > 0, L_2 > 0 \) and skew-symmetric real matrices \( T_1 \) and \( T_2 \) such that

\[
\Phi(\mu_1) + \mathcal{H}^T \begin{bmatrix} L_1 & T_1 \\ T_1^T & -L_1 \end{bmatrix} \mathcal{H} < 0
\] (31)

\[
\Phi(\mu_2) + \mathcal{H}^T \begin{bmatrix} L_2 & T_2 \\ T_2^T & -L_2 \end{bmatrix} \mathcal{H} < 0
\] (32)

which, respectively, are equivalent to (23) and (24) for \( i = 1, \) and (23) and (24) for \( i = 2 \). Thus, if the conditions in (20)–(24) are satisfied, there exists a scalar \( \varepsilon_0 > 0 \) such that \( \dot{V}(t, x_t) \leq -\varepsilon_0 \xi^T(t) \xi(t) \leq -\varepsilon_0 \alpha^T(t) x(t) \), which means that the system (14) subject to (15) is asymptotically stable.

Remark 3: The proof of Proposition 1 presents an approach to stability analysis of time-delay systems. The defining feature of it lies in that: i) A cubic matrix-valued polynomial, i.e. \( \sum_{i=0}^{3} d^i(t) P_i \) (see (19)), is introduced in the Lyapunov-Krasovskii functional; ii) A quartic matrix-valued polynomial, i.e. \( \sum_{i=0}^{4} d^i(t) \Phi_i \) (see (30)), is produced in the derivative of the Lyapunov-Krasovskii functional; and iii) Theorem 1 is employed to obtain two necessary and sufficient conditions such that \( \sum_{i=0}^{3} d^i(t) P_i > 0 \) and \( \sum_{i=0}^{4} d^i(t) \Phi_i < 0 \) for \( d(t) \in [0, \bar{h}] \), respectively. If we do not employ Theorem 1, some other methods should be used to estimate them, which definitely yields conservative stability criteria. Moreover, it should be mentioned that, to the best of the authors’ knowledge, there is no effective method available to estimate such cubic and quartic polynomials on the time-varying delay. A well-studied numerical example in Section IV shows that Proposition 1 can deliver some larger delay upper bounds than some existing ones.

B. Stability Criteria for the System (14) in Case 2

In Case 2, since information on delay-derivative is unknown, the Lyapunov-Krasovskii functional candidate is chosen as

\[
\dot{V}(t, x_t) = \tilde{\psi}_1^T(t) P \tilde{\psi}_1(t) + \dot{V}_1(t, x_t) + \dot{V}_2(t, x_t)
\] (33)

where

\[
\tilde{\psi}_1(t) = \int_{t-h}^{t} \tilde{\psi}_1(t, s) Q \tilde{\psi}_2(s, t) ds
\]

\[
\dot{V}_2(t, x_t) = \bar{h} \int_{t-h}^{t} (\bar{h} - t + s)x^T(s) R s x(s) ds
\]

with \( P > 0, Q > 0, R > 0 \); and

\[
\begin{align*}
\tilde{\psi}_1(t) &= \text{col}\{x(t), \int_{t-h}^{t} x(s) ds, \int_{t-h}^{t} (t - s)x(s) ds\} \\
\tilde{\psi}_2(s, t) &= \text{col}\{x(s), x(t), \int_{t-h}^{t} x(t) ds, \int_{t-h}^{t} x(s) ds\}
\end{align*}
\]

Proposition 2: For a given \( \bar{h} > 0 \), the system (14) with (16) is asymptotically stable if there exist real matrices \( P > 0, Q > 0, R > 0 \), symmetric real matrices \( Z_1, Z_2 \), real matrices \( Y_1, Y_2 \) with appropriate dimensions and two scalars \( \varepsilon_1 \) and \( \varepsilon_2 \) such that

\[
\begin{bmatrix} R - Z_1 & Y_1^T \\ Y_1 & R \end{bmatrix} \geq 0, \quad \begin{bmatrix} R & Y_2^T \\ Y_2 & R - Z_2 \end{bmatrix} \geq 0
\] (34)

\[
\Phi - \varepsilon_1 (\mathcal{H}_2 + \mathcal{H}_1)^T (\mathcal{H}_2 + \mathcal{H}_1) < 0
\] (35)

\[
\Phi - \varepsilon_2 (\mathcal{H}_2 - \mathcal{H}_1)^T (\mathcal{H}_2 - \mathcal{H}_1) < 0
\] (36)

where \( \mathcal{R} = \text{diag}(R, 3R, 5R, 7R, 9R) \), and

\[
\mathcal{H}_1 = \begin{bmatrix} \bar{h} I & 0 \\ 0 & \bar{h} I \end{bmatrix}, \quad \mathcal{H}_2 = \begin{bmatrix} \bar{h} I & -2I \\ 0 & \bar{h} I - 2I \end{bmatrix}
\]

\[
\Phi = \begin{bmatrix} \frac{1}{2} \Phi_0 & 0 \\ \frac{1}{2} \Phi_0 & \frac{1}{2} \Phi_3 \end{bmatrix}, \quad \Phi_0 = \begin{bmatrix} \Phi_0 & 0 \\ 0 & \frac{1}{2} \Phi_3 \end{bmatrix}, \quad \Phi_3 = \begin{bmatrix} \Phi_3 & 0 \\ 0 & \frac{1}{2} \Phi_3 \end{bmatrix}
\]

where

\[
\begin{align*}
\Phi_0 &= \bar{h}^2 C_7^T R C_0 + C_{10}^T Q C_{30} - C_{40}^T Q C_{40} - C_5^T R C_6 \\
&\quad - C_7^T (\mathcal{R} + Z_1) C_5 + \text{He}[\Sigma_{j=1}^{N_j} Q D_j]\end{align*}
\]
\[ \dot{\tilde{V}}(t, x_i) = 2\tilde{\psi}_1^T(t)P\tilde{\psi}_1(t) - \tilde{\psi}_2^T(t - \tilde{t}, t)Q\tilde{\psi}_2(t - \tilde{t}, t) + \frac{1}{\tilde{h}}\int_{t - \tilde{h}}^{t} 2\tilde{\psi}_1^T(s, t)Q\frac{\partial}{\partial t}\tilde{\psi}_2(s, t)\, ds + \tilde{h}\tilde{x}^T(s)R\tilde{x}(s)\, ds. \]  

Let \( \tilde{\xi}(t) = \text{col}([x(t), x(t - d(t)), x(t - \tilde{h}), v_1(t), v_2(t)]) \). Then

\[ \tilde{\psi}_1(t) = \tilde{C}_1 + d(t)\tilde{C}_{11} + d^2(t)\tilde{C}_{12} \]

\[ \tilde{\psi}_1(t) = \tilde{C}_20 + d(t)\tilde{C}_{21} \]

\[ \tilde{\psi}_2(t, t) = \tilde{C}_{30} + d(t)\tilde{C}_{31} + d^2(t)\tilde{C}_{32} \]

\[ \tilde{\psi}_2(t - \tilde{t}, t) = \tilde{C}_{40} + d(t)\tilde{C}_{41} + d^2(t)\tilde{C}_{42}. \]

Note that

\[ \frac{\partial}{\partial t}\tilde{\psi}_2(s, t) = \tilde{N}_1 + (t - \tilde{t} - s)\tilde{N}_2 + (t - s)\tilde{N}_3 \]

which leads to

\[ \int_{t - \tilde{h}}^{t} 2\tilde{\psi}_2^T(s, t)Q\frac{\partial}{\partial t}\tilde{\psi}_2(s, t)\, ds = 2\tilde{N}_1^T Q\int_{t - \tilde{h}}^{t} \tilde{\psi}_2(s, t)\, ds \]

\[ + 2\tilde{N}_2^T Q\int_{t - \tilde{h}}^{t} (t - \tilde{t} - s)\tilde{\psi}_2(s, t)\, ds \]

\[ + 2\tilde{N}_3^T Q\int_{t - \tilde{h}}^{t} (t - s)\tilde{\psi}_2(s, t)\, ds \]

After some algebraic manipulations, one has

\[ \int_{t - \tilde{h}}^{t} \tilde{\psi}_2(s, t)\, ds = \sum_{j=0}^{3} d^j(t)\tilde{D}_{1j} \]

\[ \int_{t - \tilde{h}}^{t} (t - \tilde{t} - s)\tilde{\psi}_2(s, t)\, ds = \sum_{j=0}^{4} d^j(t)\tilde{D}_{2j} \]

\[ \int_{t - \tilde{h}}^{t} (t - s)\tilde{\psi}_2(s, t)\, ds = \sum_{j=0}^{4} d^j(t)\tilde{D}_{3j} \]

On the other hand

\[ -\tilde{h}\int_{t - \tilde{h}}^{t} \tilde{x}^T(s)R\tilde{x}(s)\, ds \leq \tilde{\xi}^T(t)[\Psi_{11} + d(t)[\Psi_{12}](t) \tilde{\xi}(t) \]

where

\[ \Psi_{11} = -C_7^T(R + Z_1)C_5 - C_6^T R C_6 - \text{He}[C_7^T Y_2 C_6] \]

\[ \Psi_{12} = \frac{1}{\tilde{h}}[C_7^T Z_1 C_5 - C_6^T Z_2 C_6 - \text{He}[C_7^T Y_1 - Y_2] C_6] \]

To sum up, one has that

\[ \hat{V}(t, x_i) \leq \tilde{\xi}^T(t) \sum_{j=0}^{4} d^j(t)\tilde{F}_j(t) \tilde{\xi}(t) \]  

where \( \tilde{F}_j(t) = \frac{1}{4} \sum_{j=0}^{4} d^j(t) \Phi_j(t) \).
task to express explicitly such a sixth-degree polynomial.

IV. NUMERICAL EXAMPLE

In this section, we take a well-studied example to compare Propositions 1 and 2 with some existing stability criteria recently reported.

Example 1: Consider the system (14) with

\[
A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}
\] (39)

Case 1: Suppose that the time-varying delay satisfies (15).

In this case, we calculate the maximum admissible upper bound of \( \bar{h} \) for \( \mu = -\mu_1 = \mu_2 \in \{0.1, 0.5, 0.8\} \). Table I lists the obtained results using some existing methods [23, Theorem 1], [24, Theorem 1], [19, Theorem 8], [13, Proposition 2] and [18, Corollary 2]. However, applying Proposition 1 in this paper gives much larger upper bounds of \( \bar{h} \), which can be seen in Table 1.

On the other hand, Table I also lists the number of decision variables (DVs) required in those methods. It is clear to see that Proposition 1 requires a larger number of DVs, which means that solving the matrix inequalities in Proposition 1 is much time-consuming. However, with the rapid development of computer technology, such a number of DVs is not a problem for high performance computers to solve the matrix inequalities in Proposition 1.

Case 2: Suppose that the time-varying delay satisfies (16).

In this case, we use Proposition 2 to compare with some existing methods [11], [25]–[27]. Table II lists both the obtained delay upper bounds and the required number of DVs by [26, Corollary 1], [27, Proposition 6], [25, Theorem 2], [13, Proposition 6] and Proposition 2 in this paper. From Table II, one can see that Proposition 2 delivers a larger upper bound \( \bar{h} \) than those in [11], [25]–[27]. Moreover, the number of DVs required in Proposition 2 is less than those of [25, Theorem 2] and [13, Proposition 6].

V. CONCLUSION

Stability of linear systems with a time-varying delay has been studied. First, a necessary and sufficient condition on matrix-valued polynomial inequalities has been established. Then, this condition has been employed to formulate two stability criteria for two cases of the time-varying delay, respectively, where the time-varying delay is differentiable or only continuous. Simulation has shown that the obtained stability criteria can provide larger delay upper bounds than some existing ones.

REFERENCES


TABLE I

THE MAXIMUM ADMISSIBLE UPPER BOUND \( \bar{h} \) FOR DIFFERENT VALUES OF \( \mu = -\mu_1 = \mu_2 \)

<table>
<thead>
<tr>
<th>Method ( \mu )</th>
<th>0.1</th>
<th>0.5</th>
<th>0.8</th>
<th>Number of DVs</th>
</tr>
</thead>
<tbody>
<tr>
<td>[22, Proposition 1]</td>
<td>4.910</td>
<td>3.233</td>
<td>2.789</td>
<td>54.5n^2 + 6.5n</td>
</tr>
<tr>
<td>[23, Theorem 1]</td>
<td>4.942</td>
<td>3.309</td>
<td>2.882</td>
<td>108n^2 + 12n</td>
</tr>
<tr>
<td>[24, Theorem 1]</td>
<td>4.996</td>
<td>3.251</td>
<td>2.867</td>
<td>38n^2 + 9n</td>
</tr>
<tr>
<td>[19, Theorem 8]</td>
<td>5.01</td>
<td>3.19</td>
<td>2.70</td>
<td>146.5n^2 + 9.5n</td>
</tr>
<tr>
<td>[13, Proposition 2]</td>
<td>4.929</td>
<td>3.252</td>
<td>2.823</td>
<td>216n^2 + 11n</td>
</tr>
<tr>
<td>[18, Corollary 2]</td>
<td>5.044</td>
<td>3.443</td>
<td>2.983</td>
<td>235n^2 + 34n</td>
</tr>
<tr>
<td>Proposition 1</td>
<td>5.147</td>
<td>3.673</td>
<td>3.258</td>
<td>367n^2 + 35n + 6</td>
</tr>
</tbody>
</table>

TABLE II

THE MAXIMUM ADMISSIBLE UPPER BOUND \( \bar{h} \) FOR CASE 2

<table>
<thead>
<tr>
<th>Method</th>
<th>( \bar{h} )</th>
<th>Number of DVs</th>
</tr>
</thead>
<tbody>
<tr>
<td>[26, Corollary 1]</td>
<td>2.18</td>
<td>54.5n^2 + 9.5n</td>
</tr>
<tr>
<td>[27, Proposition 1]</td>
<td>2.18</td>
<td>54n^2 + 9n</td>
</tr>
<tr>
<td>[25, Theorem 2]</td>
<td>2.39</td>
<td>154.5n^2 + 4.5n</td>
</tr>
<tr>
<td>[13, Proposition 6]</td>
<td>2.53</td>
<td>119.5n^2 + 3.5n</td>
</tr>
<tr>
<td>Proposition 2</td>
<td>3.04</td>
<td>98n^2 + 10n + 2</td>
</tr>
</tbody>
</table>
Xian-Ming Zhang (M’16–SM’18) received the M.Sc. degree in applied mathematics and the Ph.D. degree in control theory and engineering from Central South University, Changsha, China, in 1992 and 2006, respectively.

In 1992, he joined Central South University, where he was an Associate Professor with the School of Mathematics and Statistics. From 2007 to 2014, he was a Post-Doctoral Research Fellow and a Lecturer with the School of Engineering and Technology, Central Queensland University, Rockhampton, QLD, Australia. From 2014 to 2016, he was a Lecturer with the Griffith School of Engineering, Griffith University, Gold Coast, QLD, Australia. In 2016, he joined the Swinburne University of Technology, Melbourne, VIC, Australia, where he is currently an Associate Professor with the School of Software and Electrical Engineering. His current research interests include H-infinity filtering, event-triggered control systems, networked control systems, neural networks, distributed systems, and time-delay systems.

Dr. Zhang was a recipient of second National Natural Science Award in China in 2013, and first Hunan Provincial Natural Science Award in Hunan Province in China in 2011, both jointly with Prof. M. Wu and Prof. Y. He, and the IET Premium Award in 2016, jointly with Prof. Q.-L. Han. He is an Associate Editor of the IEEE Transactions on Cybernetics, Neural Processing Letters, Journal of the Franklin Institute, International Journal of Control, Automation, and Systems and the Neurocomputing, and he is an Editorial Board Member of Neural Computing and Applications.

Xiaohua Ge (M’18) received the B.Eng. degree in electronics and information engineering from Nanchang Hangkong University, Nanchang, China, in 2008, the M.Eng. degree in control theory and control engineering from Hangzhou Dianzi University, Hangzhou, China, in 2011, and the Ph.D. degree in computer engineering from Central Queensland University, Rockhampton, QLD, Australia, in 2014.

From 2011 to 2013, he was a Research Assistant with the Centre for Intelligent and Networked Systems, Central Queensland University, Rockhampton, QLD, Australia, where he was a Research Fellow in 2014. From 2015 to 2017, he was a Research Fellow with the Griffith School of Engineering, Griffith University, Gold Coast, QLD, Australia. He is currently a Senior Lecturer with the School of Software and Electrical Engineering, Swinburne University of Technology, Melbourne, VIC, Australia.

His research interests include distributed estimation over sensor networks, distributed coordination in multi-agent systems, security and privacy preserving in cyber-physical systems.

Qing-Long Han (M’09–SM’13–F’19) received the B.Sc. degree in Mathematics from Shandong Normal University, Jinan, China, in 1983, and the M.Sc. and Ph.D. degrees in Control Engineering and Electrical Engineering from East China University of Science and Technology, Shanghai, China, in 1992 and 1997, respectively.

From September 1997 to December 1998, he was a Post-doctoral Research Fellow with the Laboratoire d’Automatique et d’Informatique Industrielle (currently, Laboratoire d’Informatique et d’Automatique pour les Systèmes), École Supérieure d’Ingénieurs de Poitiers (currently, Ecole Nationale Supérieure d’Ingénieurs de Poitiers), Université de Poitiers, France. From January 1999 to August 2001, he was a Research Assistant Professor with the Department of Mechanical and Industrial Engineering at Southern Illinois University at Edwardsville, USA. From September 2001 to December 2014, he was a Laureate Professor, an Associate Dean (Research and Innovation) with the Higher Education Division, and the Founding Director of the Centre for Intelligent and Networked Systems at Central Queensland University, Australia. From December 2014 to May 2016, he was Deputy Dean (Research), with the Griffith Sciences, and a Professor with the Griffith School of Engineering, Griffith University, Australia. In May 2016, he joined Swinburne University of Technology, Australia, where he is currently Pro Vice-Chancellor (Research Quality) and a Distinguished Professor. His research interests include networked control systems, multi-agent systems, time-delay systems, complex dynamical systems and neural networks.

Professor Han is a Highly Cited Researcher according to Clarivate Analytics (formerly Thomson Reuters). He is a Fellow of The Institution of Engineers Australia. He is an Associate Editor of several international journals, including the IEEE Transactions on Cybernetics, the IEEE Transactions on Industrial Electronics, the IEEE Transactions on Industrial Informatics, IEEE Industrial Electronics Magazine, the IEEE/CNS Journal of Automatica Sinica, Control Engineering Practice, and Information Sciences.