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A novel matrix approach for the stability and stabilization analysis of colored Petri nets

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Abstract In this study, the stability and stabilization problem of a colored Petri net based on the semi-tensor product of matrices is investigated. First, the marking evolution equation of the colored Petri net in a Boolean algebra framework is established, and the necessary and sufficient condition for the stability of the equilibrium point of the colored Petri net is given. Then, the concept of the pre-k steps reachability set is defined and is used to study the problem of marking feedback stabilization. Some properties of the pre-k steps reachability set are developed. The condition of the stabilization of the colored Petri net is given. The algorithm of the optimal marking feedback controller is designed. The proposed method in this paper could judge the stability and stabilization of the colored Petri net by matrix approach. The obtained results are simple and easy to implement by computer. An example is provided to illustrate the effectiveness of the proposed method.

Keywords discrete event system, colored Petri net, semi-tensor product of matrices, stability, equilibrium point, stabilization

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1 Introduction

A Petri net is a kind of mathematical model and tool for the modelling and analysis of distributed systems. It is especially convenient for describing the relationships of the sequence, concurrency, conflict, and synchronization of the system process or component. It is a powerful and efficient formal method for simulating, analyzing, and controlling discrete event systems (DESs). It supports a wide range of interactive expressions, has intuitive graphic representation, and is relatively simple to implement. For a review of the history of Petri nets and an extensive bibliography, the reader can refer to [1]. The colored Petri net puts different colors on the tokes of the net. The aim of this method is to classify the tokes so that we can fold the net. The colored Petri net is an advanced high-level Petri net. The main difference between the colored Petri net and an ordinary Petri net is that, for the colored Petri net, a more compact representation is achieved by equipping each token with an attached data value called the token color that may be of an arbitrarily complex type. Therefore, it can describe complex systems in a manageable way. Although this kind of net does not have stronger simulation ability than an ordinary Petri net, its main merit model is that it makes the modeling of complex systems clearer and simpler. For a detailed introduction to the concept, definition, analysis methods, and various applications of colored Petri nets, please refer to [2–4].

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Classical control theory studies the time driven system, which is simulated by a differential equation or differential equations. However, many complex dynamic systems in engineering and social fields are discrete event driven. DESs are event-driven dynamical systems whose state transition mechanism is triggered by the instantaneous occurrence of discrete events; such behavior can be found in many large and complex systems, such as computer and communication networks, flexible manufacturing systems, and intelligent transportation systems and queueing systems. DESs follow intricate man-made rules instead of physical laws. As a result, performance analysis and control synthesis of DESs become very difficult because their state evolution processes are not described by conventional differential and difference equations. Petri nets constitute a well-known formal paradigm for the modeling, analysis, and control of DESs. They have received wide attention within the automatic control community according to the difference of the system model. This approach has been adopted by many researchers, and it has been proved to be very effective and useful. Thus, it has attracted the attention of many researches in the Petri net field. Yamalidou et al. [5] studied the feedback control problem of a Petri net based on place invariants in reference. In addition, other researchers have studied the properties of special Petri nets. For example, Mahulea et al. [6, 7] analyzed the observability problem of continuous Petri nets, and Vázquez et al. [8] studied the controllability of timed continuous Petri nets.

Many control problems of ordinary Petri nets can be extended to colored Petri nets. Giua and Seatzu [9] presented control problems of ordinary Petri nets, which also exist in the colored Petri net. Zhao et al. [10] modeled the colored Petri net by using an algebraic method and studied its reachability and controllability, but so far there are no related studies on other control issues, such as stability, stabilization, and observability.

In recent years, a new matrix product called semi-tensor product of matrices has been proposed by Cheng [11]. It generalizes the ordinary matrix product to the case where the column number of the fore matrix is different from the row number of the hind matrix. The generalized product not only keeps the main properties of the original matrix product, but also has a better pseudo commutativity property than that before the generalization. Therefore, it is a convenient and powerful mathematical tool. By using the semi-tensor product of the matrix, a DES can be transformed into a linear algebraic equation form, so that the classical control theory and method can be used to analyze and control the DES. The semi-tensor product of matrix has been widely used in Boolean networks [12, 13], finite automata [14, 15], network game theory [16–18] and other fields, and it has solved many logic dynamic system control problems that were difficult to solve before. In the Petri net field, the semi-tensor product of matrices has obvious advantages in the representation of the structure properties of the Petri net. Han et al. [19] used the semi-tensor product of matrices to calculate the siphon of Petri nets. Han et al. [20] used the semi-tensor product of matrices to described the dynamic behavior of a Petri net and obtained a discrete-time linear equation. Then, the Petri net was analyzed by using the traditional analysis method. Some conditions for reachability and controllability of ordinary Petri net were obtained.

In this study, we investigate the stability and stabilization analysis of a colored Petri net based on the semi-tensor product of matrices. The major novel contributions of this work are as follows.

(1) Definitions of the equilibrium point, k-steps pre-reachability set, stability of equilibrium point, and stabilizability of a colored Petri net are determined.

(2) The marking evolution equation of colored Petri nets in the framework of Boolean algebra is developed. By using the obtained marking evolution equation, the condition for the stability of colored Petri nets is obtained.

(3) In order to study the stabilizability problem of colored Petri net, the concept of the k-step pre-reachability set of the colored Petri net is given, and the condition of the stabilizability of the equilibrium point is also obtained.

(4) By using the marking evolution equation and k-steps pre-reachability set, an algorithm is designed to calculate the marking feedback controller.

2 Preliminaries

2.1 Notations

- $\mathbb{N}$ is a set of natural numbers.
- $|X|$ is the potential of a set.
- $\mathcal{N}^+$ is a set of positive integers.
- $\mathbb{R}^n$ is a set of all vectors of dimension n.
- $\mathcal{M}_{m\times n}$ is a set of $m\times n$ real matrices.
- $M_{(i,j)}$ denotes the (i,j) element of matrix M.
- $\mathrm{Row}_i(M)$ is the i-th row of matrix M.
- $\mathrm{Col}_i(M)$ is the i-th column of matrix M.
- $\mathrm{Col}(M)$ is the set of all columns of matrix M.
- $\mathcal{D} := \{0,1\}$.
- If $B\in\mathcal{M}_{m\times n}$ and $B_{(i,j)}\in\mathcal{D}$, $\forall i\in\{1,2,\ldots,m\}$, $j\in\{1,2,\ldots,n\}$, then B is called the Boolean matrix.
- $\mathcal{B}_{m\times n}$ is a set of $m\times n$ Boolean matrices.
- $1_k := [\underbrace{1,1,\ldots,1}_{k}]$, $\delta_n^0 := [\underbrace{0,0,\ldots,0}_{n}]^{\mathrm{T}}$.
- δ_n^k denotes the k-th column of I_n, $1\leqslant k\leqslant n$.
- $\Delta_n := \{\delta_n^1,\delta_n^2,\ldots,\delta_n^n\}$, $\widetilde{\Delta}_n := \{\delta_n^0\}\bigcup\Delta_n$.
- $L\in\mathcal{M}_{m\times n}$ is a logical matrix; if $\mathrm{Col}(L)\subseteq\Delta_m$, $\mathcal{L}_{m\times n}$ denotes the set of $m\times n$ logical matrices.
- $L\in\mathcal{M}_{m\times n}$ is a generalized logical matrix; if $\mathrm{Col}(L)\subseteq\tilde{\Delta}_m$, $\tilde{\mathcal{L}}_{m\times n}$ denotes the set of $m\times n$ generalized logical matrix.
- If $L\in\tilde{\mathcal{L}}_{m\times n}$, then it can be expressed as $L=[\delta_m^{i_1},\delta_m^{i_2},\ldots,\delta_m^{i_n}]$, and it is briefly denoted by $L=\delta_m[i_1,i_2,\ldots,i_n]$, where $i_k\in\{0,1,\ldots,m\}$, $1\leqslant k\leqslant n$.
- If $A\in\tilde{\mathcal{L}}_{n\times m}$, $\Xi(A) := \{i|\mathrm{Col}_i(A)\in\Delta_n\}$.
- If $A=(a_{ij})_{m\times n}\in\mathcal{B}_{m\times n}$, $B=(b_{ij})_{m\times n}\in\mathcal{B}_{m\times n}$, then the matrix $A\vee B := ((a_{ij})\vee(b_{ij}))_{m\times n}$, $A\wedge B := ((a_{ij})\wedge(b_{ij}))_{m\times n}$, where $\vee$ and $\wedge$ are the logical disjunctive operation and logical conjunctive operation, respectively.
- If $A\in\mathcal{B}_{m\times n}$, $\Theta(\mathbb{A}) := \{A\in\mathcal{L}_{m\times n}|A\wedge\mathbb{A}=A\}$, then $x\in\mathcal{B}_{m\times 1}$, $\Theta(x) := \{y\in\triangle_m|y\wedge x=y\}$.

2.2 Semi-tensor product (STP) of matrices

We give some definitions for the semi-tensor product of matrices, which will be used in the sequel.

Definition 1 ([11]). Given $A\in\mathcal{M}_{m\times n}$, $B\in\mathcal{M}_{p\times q}$, the semi-tensor product of A and B is defined as

$$A\ltimes B=(A\otimes I_{t/n})(B\otimes I_{t/p}),$$

where $t=\mathrm{lcm}(n,p)$ is the least common multiple of n and p, and $\otimes$ is the Kronecker product.

Remark 1. When $n=p$, $A\ltimes B=AB$. Therefore, the STP is a generalization of the conventional matrix product.

Definition 2 ([11]). A swap matrix $W_{[m,n]}$ is an $mn\times mn$ matrix, which is defined as

$$W_{[m,n]}=\delta_{mn}[1,m+1,2m+1,\ldots,(n-1)m+1,2,m+2,2m+2,\ldots,(n-1)m+2,\ldots,m,2m,3m,\ldots,nm].$$

Lemma 1 ([11]). Let $X\in\mathbb{R}^m$ and $Y\in\mathbb{R}^n$ be two column vectors. Then

$$W_{[m,n]}XY=YX,\quad W_{[n,m]}YX=XY.$$

2.3 Boolean algebra

Definition 3 ([21]). Given $\alpha, \beta \in \mathcal{D}$, the Boolean addition and Boolean multiplication of α and β are defined as

$$\alpha +_{\mathcal{B}} \beta := \alpha \vee \beta, \quad \alpha \times_{\mathcal{B}} \beta := \alpha \wedge \beta,$$

where $\vee$ and $\wedge$ are the logical disjunctive operation and logical conjunctive operation, respectively.

$\{\mathcal{D}, +_{\mathcal{B}}, \times_{\mathcal{B}}\}$ forms a algebraic system that we call Boolean algebra. Then we can define the Boolean addition and Boolean multiplication.

Definition 4 ([21]). Assume $A = (a_{ij})_{m\times n} \in \mathcal{B}_{m\times n}$, $B = (b_{ij})_{m\times n} \in \mathcal{B}_{m\times n}$. Then

$$A +_{\mathcal{B}} B = (a_{ij} +_{\mathcal{B}} b_{ij})_{m\times n} \in \mathcal{B}_{m\times n}.$$

Definition 5 ([21]). Assume $A = (a_{ij})_{m\times n} \in \mathcal{B}_{m\times n}$, $B = (b_{ij})_{n\times s} \in \mathcal{B}_{n\times s}$. Then

$$A \ltimes_{\mathcal{B}} B := C = (c_{ij})_{m\times s} \in \mathcal{B}_{m\times s},$$

where $c_{ij} = (a_{i1} \times_{\mathcal{B}} b_{1j}) +_{\mathcal{B}} (a_{i2} \times_{\mathcal{B}} b_{2j}) +_{\mathcal{B}} \cdots +_{\mathcal{B}} (a_{in} \times_{\mathcal{B}} b_{nj})$. In particular, the Boolean power of $A \in \mathcal{B}_{n\times n}$ can be expressed as

$$A^{(k)} := \underbrace{A \ltimes_{\mathcal{B}} A \ltimes_{\mathcal{B}} \cdots \ltimes_{\mathcal{B}} A}_{k}, \quad \forall k \in \mathcal{N}^{+}.$$

2.4 Colored Petri net (CPN)

In this subsection, we introduce some concepts about the colored Petri net [2].

The function $m \in [S \to \mathbb{N}]$ is a multi-set m over a non-empty set S. The number of appearances of the element s in the multi-set m is denoted by a non-negative integer $m(s) \in \mathbb{N}$. The formal sum: $\sum_{s\in S} m(s)'s$ represents the multi-set m. The S_{MS} represents the set of all multi-sets over S. The expression is one kind of colored Petri nets representation. The expression representation uses arcs and guards, while the function representation uses linear functions between multi-sets.

Before giving the abstract definition of a colored Petri net, we fix the concrete syntax in which the modeler writes the net expressions.

- The type(v) represents the type of a variable v.
- Type(expr) represents the type of an expression expr.
- Var(expr) represents the set of variables in an expression expr.
- A binding of a set of variables: V associates each variable $v \in V$ with an element $b(v) \in \mathrm{Type}(v)$. Var(expr) is required to be a subset of the variables of b; the value of variable v can be obtained by substituting for each variable $v \in \mathrm{Var(expr)}$ the value $b(v) \in \mathrm{Type}(v)$, which is determined by the binding.

A colored Petri net is a tuple $\mathrm{CPN} = (\Sigma, P, T, A, N, C, G, E, I)$ satisfying the following conditions:

(i) Color sets Σ is a finite set of non-empty types.

(ii) P represents a finite set of places.

(iii) T represents a finite set of transitions.

(iv) A represents a finite set of arcs such that: $P \bigcap T = P \bigcap A = T \bigcap A = \emptyset$.

(v) N represents a node function. It is defined from A into $P \times T \cup T \times P$.

(vi) C represents a color function. It is defined from P into Σ.

(vii) G represents a guard function. It is defined from T into expressions such that $\forall t \in T$: $[\mathrm{Type}(G(t)) = B(t) \wedge \mathrm{Type}(\mathrm{Var}(G(t))) \subseteq \Sigma]$.

(viii) E represents an arc expression function. It is defined from A into expressions such that $\forall a \in A$: $[\mathrm{Type}(E(a)) = C(p(a))_{\mathrm{MS}} \wedge \mathrm{Type}(\mathrm{Var}(E(a))) \subseteq \Sigma]$, where $p(a)$ is the place of $N(a)$.

(ix) I represents an initialization function. It is defined from P into closed expressions such that: $\forall p \in P : [\mathrm{Type}(I(p)) = C(p)_{\mathrm{MS}}]$.

A binding of a transition t is a function b defined on $\mathrm{Var}(t)$, such that (i) $\forall v \in \mathrm{Var}(t) : b(v) \in \mathrm{Type}(v)$. (ii) $G(t)\langle b\rangle$. $B(t)$ represents the set of all bindings for t. A place instance is a pair (p, c), where $p \in P$ and $c \in C(p)$. A transition instance is a pair (t, b), where $t \in T$ and $b \in B(t)$. TE represents the set of all place instances. BE represents the set of all transition instances. A marking is a multi-set over TE. A step is a non-empty and finite multi-set over BE. The sets of all markings and steps are denoted by $\mathbb{M}$ and $\mathbb{Y}$, respectively. A step Y is enabled in a marking M if the following property is satisfied: $\forall p \in P : \sum_{(t,b)\in\mathbb{Y}} E(p, t)\langle b\rangle \leqslant M(p)$. When a step Y is enabled in a marking M_1, it may occur, changing the marking M_1 to another marking M_2, which is defined by $\forall p \in P : M_2(p) = (M_1(p) - \sum_{(t,b)\in\mathbb{Y}} E(p, t)\langle b\rangle) + \sum_{(t,b)\in\mathbb{Y}} E(t, p)\langle b\rangle$. The first sum is called the removed tokens and the second is called the added tokens. If M_2 is directly reachable from M_1 by the occurrence of the step Y, we mark it as $M_1[Y > M_2$. We call a transition and a binding for all variables a control quantity. In order to obtain a better understanding of how a colored Petri net runs, please refer to example 1 in [10], which is a colored Petri net about dining philosophers.

Lemma 2 ([10])**.** Assume the number of the markings of the state space is finite. Then the marking evolution equation of $\langle \mathrm{CPN}, M_0\rangle$ could be expressed as the following linear form:

$$x(k + 1) = Lu(k)x(k),$$

where $L \in \tilde{\mathcal{L}}_{s\times sr}$ is the control quantity marking transfer matrix; $u(k)$ and $x(k)$ are the vector of the control quantity and marking at the k-th step, respectively; r is the number of control quantities; and s is the number of markings.

3 The stability of the colored Petri net

Definition 6. Assume M is a marking of the colored Petri net $\langle \mathrm{CPN}, M_0\rangle$, that is $M \in R(\mathrm{CPN}, M_0)$. If there exists an enable transition $t \in T$, and $M[t > M$, we call M an equilibrium point of $\langle \mathrm{CPN}, M_0\rangle$.

Definition 7. M_p is an equilibrium point of $\langle \mathrm{CPN}, M_0\rangle$, and M is a marking of $\langle \mathrm{CPN}, M_0\rangle$. If all the marking traces starting from M converge to M_p after finite steps, then the marking M is stable at the equilibrium point M_p. If every marking of $\langle \mathrm{CPN}, M_0\rangle$ is stable at M_p, then $\langle \mathrm{CPN}, M_0\rangle$ is stable at the equilibrium point M_p.

Let $X(M, \delta, k)$ be the set of markings that are reached after k steps from M, and $X(M, \delta, 0) = M$, where δ is the firing sequence of length k.

From the above definition, the marking M is stable at the equilibrium point M_p if and only if there exists $T(M) \in \mathcal{N}^+$, such that $X(M, \delta, k) = \{M_p\}$, $\forall k \geqslant T(M)$. Let $T_p(M)$ be the smallest positive integer that makes the above formula hold; we call $T_p(M)$ the transient period of M. If the transient period of the colored Petri net system is defined as $T_p := \max_{M\in R(\mathrm{CPN},M_0)} T_p(M)$, then we can conclude that $T_p < |R(\mathrm{CPN}, M_0)|$.

Let $\Gamma(M)$ be the set of enable control quantity under marking M. We then have Lemma 3.

Lemma 3. Assume the marking evolution equation of $\langle \mathrm{CPN}, M_0\rangle$ is $x(k + 1) = Lu(k)x(k)$, $M := \delta_s^i \in R(\mathrm{CPN}, M_0)$ is a marking of $\langle \mathrm{CPN}, M_0\rangle$, and $\Xi(LW_{[s,m]}\delta_s^i) = \{i_1, i_2, \ldots, i_q\}$. Then

$$\Gamma(M) = \Gamma(\delta_s^i) = \{\delta_m^{i_1}, \delta_m^{i_2}, \ldots, \delta_m^{i_q}\},$$

where $s = |R(\mathrm{CPN}, M_0)|$, and $\delta_m^{i_j}$ is the vector form of the control quantity t_{i_j}, $1 \leqslant j \leqslant q$.

In fact, $\Gamma(M)$ is a set value functions from M to 2^T, that is, $\Gamma : M \to 2^T$, where 2^T is the power set of the control quantity T. By resorting to the semi-tensor product of matrices, $\Gamma(M)$ could be obtained in the following way: $\tilde{u}(k) = Hx(k)$, where $x(k)$ is the vector form of the marking of step k.

$\tilde{u}(k) = (u^1(k), u^2(k), \ldots, u^r(k))^{\mathrm{T}}$ is the vector form of set function Γ at step k. $u^j(k) = 1$ indicates that the control quantity t_j is enabled at marking $x(k)$; otherwise $u^j(k) = 0$. $H \in \mathcal{B}_{m\times n}$ is called the control quantity enable matrix of the colored Petri net $\langle \mathrm{CPN}, M_0\rangle$. We can easily obtain Lemma 4.

Lemma 4. Assume the marking evolution equation of $\langle \mathrm{CPN}, M_0\rangle$ is $x(k + 1) = Lu(k)x(k)$ and $M := \delta_s^i \in R(\mathrm{CPN}, M_0)$ is a marking of $\langle \mathrm{CPN}, M_0\rangle$. Then

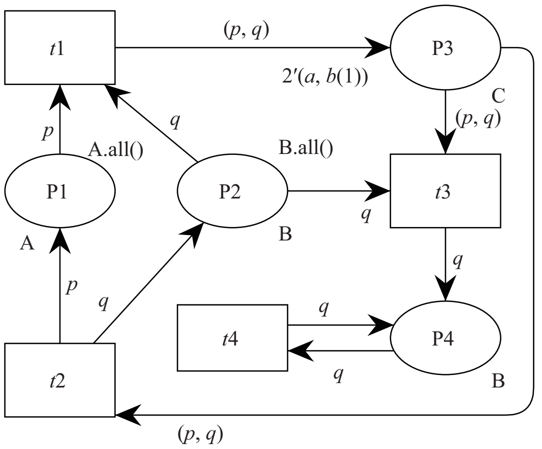

Figure 1 The colored Petri net.

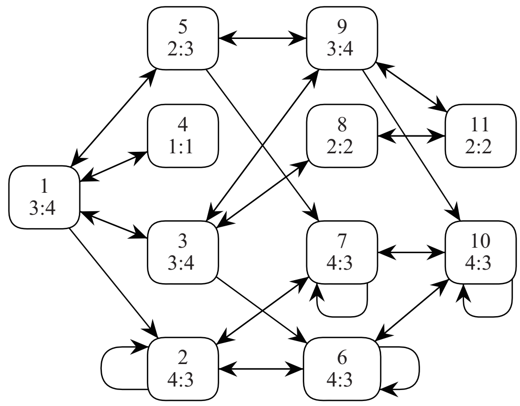

Figure 2 The state space of the colored Petri net.

$$\Gamma(M) = \Gamma(\delta_s^i) = \Theta(\mathrm{Col}_i(H)), \quad 1 \leqslant i \leqslant s.$$

Example 1. The colored Petri net and its state space form can be seen in Figures 1 and 2. Following is the definition:

val $n = 2$;

colset A = with a;

colset B = index b with $1, \ldots, n$;

colset C = product A×B;

var p : A;

var q : B.

The markings of the state space are as follows:

[1] P1 : $1'a$, P2 : $1'b(1) + +1'b(2)$, P3 : $2'(a, b(1))$, P4 : empty;

[2] P1 : $1'a$, P2 : $1'b(2)$, P3 : $1'(a, b(1))$, P4 : $1'b(1)$;

[3] P1 : $2'a$, P2 : $2'b(1) + +1'b(2)$, P3 : $1'(a, b(1))$, P4 : empty;

[4] P1 : empty, P2 : $1'b(2)$, P3 : $3'(a, b(1))$, P4 : empty;

[5] P1 : empty, P2 : $1'b(1)$, P3 : $2'(a, b(1)) + +1'(a, b(2))$, P4 : empty;

[6] P1 : $2'a$, P2 : $1'b(1) + +1'b(2)$, P3 : empty, P4 : $1'b(1)$;

[7] P1 : empty, P2 : empty, P3 : $1'(a, b(1)) + +1'(a, b(2))$, P4 : $1'b(1)$;

[8] P1 : $3'a$, P2 : $3'b(1) + +1'b(2)$, P3 : empty, P4 : empty;

[9] P1 : $1'a$, P2 : $2'b(1)$, P3 : $1'(a, b(1)) + +1'(a, b(2))$, P4 : empty;

[10] P1 : $1'a$, P2 : $1'b(1)$, P3 : $1'(a, b(2))$, P4 : $1'b(1)$;

[11] P1 : $2'a$, P2 : $3'b(1)$, P3 : $1'(a, b(2))$, P4 : empty.

The manners how the state transfers from one to another:

[1] $1 \to 2$ $t3$ $q = b(1)$ $p = a$ $u = 5$, **[2]** $1 \to 3$ $t2$ $q = b(1)$ $p = a$ $u = 3$,
[3] $1 \to 4$ $t1$ $q = b(1)$ $p = a$ $u = 1$, **[4]** $1 \to 5$ $t1$ $q = b(2)$ $p = a$ $u = 2$,
[5] $2 \to 2$ $t4$ $q = b(1)$ $u = 7$, **[6]** $2 \to 6$ $t2$ $q = b(1)$ $p = a$ $u = 3$,
[7] $2 \to 7$ $t1$ $q = b(2)$ $p = a$ $u = 2$, **[8]** $3 \to 6$ $t3$ $q = b(1)$ $p = a$ $u = 5$,
[9] $3 \to 8$ $t2$ $q = b(1)$ $p = a$ $u = 3$, **[10]** $3 \to 1$ $t1$ $q = b(1)$ $p = a$ $u = 1$,
[11] $3 \to 9$ $t1$ $q = b(2)$ $p = a$ $u = 2$, **[12]** $4 \to 1$ $t2$ $q = b(1)$ $p = a$ $u = 3$,
[13] $5 \to 7$ $t3$ $q = b(1)$ $p = a$ $u = 5$, **[14]** $5 \to 1$ $t2$ $q = b(2)$ $p = a$ $u = 4$,
[15] $5 \to 9$ $t2$ $q = b(1)$ $p = a$ $u = 3$, **[16]** $6 \to 6$ $t4$ $q = b(1)$ $u = 7$,
[17] $6 \to 2$ $t1$ $q = b(1)$ $p = a$ $u = 1$, **[18]** $6 \to 10$ $t1$ $q = b(2)$ $p = a$ $u = 2$,
[19] $7 \to 7$ $t4$ $q = b(1)$ $u = 7$, **[20]** $7 \to 2$ $t2$ $q = b(2)$ $p = a$ $u = 4$,
[21] $7 \to 10$ $t2$ $q = b(1)$ $p = a$ $u = 3$, **[22]** $8 \to 11$ $t1$ $q = b(2)$ $p = a$ $u = 2$,
[23] $8 \to 3$ $t1$ $q = b(1)$ $p = a$ $u = 1$, **[24]** $9 \to 10$ $t3$ $q = b(1)$ $p = a$ $u = 5$,
[25] $9 \to 11$ $t2$ $q = b(1)$ $p = a$ $u = 3$, **[26]** $9 \to 3$ $t2$ $q = b(2)$ $p = a$ $u = 4$,
[27] $9 \to 5$ $t1$ $q = b(1)$ $p = a$ $u = 1$, **[28]** $10 \to 10$ $t4$ $q = b(1)$ $u = 7$,
[29] $10 \to 6$ $t2$ $q = b(2)$ $p = a$ $u = 4$, **[30]** $10 \to 7$ $t1$ $q = b(1)$ $p = a$ $u = 1$,
[31] $11 \to 8$ $t2$ $q = b(2)$ $p = a$ $u = 4$, **[32]** $11 \to 9$ $t1$ $q = b(1)$ $p = a$ $u = 1$.

The reachable set of this colored Petri net is $R(\mathrm{CPN}, M_1) = \{M_1, M_2, M_3, \ldots, M_{11}\}$. Let $M_i \sim \delta_{11}^i$, $1 \leqslant i \leqslant 11$; $t_j \sim \delta_4^j$, $1 \leqslant j \leqslant 4$; $q(k) = \delta_2^k$, $k = 1, 2$. Then the control quantity can be expressed as $u = \delta_4^j \ltimes \delta_2^k$, $1 \leqslant j \leqslant 4$, $k = 1, 2$. The marking evolution equation of the colored Petri net is

$$x(k+1) = Lu(k)x(k),$$
$$L_1 = \delta_{11}[4\ 0\ 1\ 0\ 0\ 2\ 0\ 3\ 5\ 7\ 9], \quad L_2 = \delta_{11}[5\ 7\ 9\ 0\ 0\ 10\ 0\ 11\ 0\ 0\ 0],$$
$$L_3 = \delta_{11}[3\ 6\ 8\ 1\ 9\ 0\ 10\ 0\ 11\ 0\ 0], \quad L_4 = \delta_{11}[0\ 0\ 0\ 0\ 1\ 0\ 2\ 0\ 3\ 6\ 8],$$
$$L_5 = \delta_{11}[2\ 0\ 6\ 0\ 7\ 0\ 0\ 0\ 10\ 0\ 0], \quad L_6 = \delta_{11}[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0],$$
$$L_7 = \delta_{11}[0\ 2\ 0\ 0\ 0\ 6\ 7\ 0\ 0\ 10\ 0], \quad L_8 = \delta_{11}[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0],$$
$$L = [L_1\ L_2\ L_3\ L_4\ L_5\ L_6\ L_7\ L_8],$$

$L = \delta_{11}[4\ 0\ 1\ 0\ 0\ 2\ 0\ 3\ 5\ 7\ 9\ 5\ 7\ 9\ 0\ 0\ 10\ 0\ 11\ 0\ 0\ 0\ 3\ 6\ 8\ 1\ 9\ 0\ 10\ 0\ 11\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 2\ 0\ 3\ 6\ 8\ 2\ 0\ 6\ 0\ 7\ 0\ 0\ 0\ 10\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 2\ 0\ 0\ 0\ 6\ 7\ 0\ 0\ 10\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]$, $u(k) = Hx(k)$, and the control quantity enable matrix is

$$H = \begin{bmatrix} 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 1 \\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 0 \\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 0 \\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1 \\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \end{bmatrix}.$$

Assume the current marking is $M_6 = \delta_{11}^6$, $\Gamma(M_6) = \Theta(\mathrm{Col}_6(H)) = \{\delta_8^1, \delta_8^2, \delta_8^7\}$, $\delta_8^1 = \delta_4^1 \ltimes \delta_2^1$, $\delta_8^2 = \delta_4^1 \ltimes \delta_2^2$, $\delta_8^7 = \delta_4^4 \ltimes \delta_2^1$. Then when the current marking is M_6, the transition is t_1, q is b_1 or b_2, or the transition is t_4, q is b_1, and M_6 could transfer to other marking.

The marking evolution equation can be transformed into the following form:

$$x(k+1) = Lu(k)x(k) = LHx^2(k) = LH\Phi_n x(k).$$

Let $A := LH\Phi_n$, $L = [\mathrm{Blk}_1(L), \mathrm{Blk}_2(L), \ldots, \mathrm{Blk}_r(L)]$, where $\mathrm{Blk}_j(L) \in \mathcal{B}_{n\times n}$ is the j-th sub block of L and $1 \leqslant j \leqslant r$. According to the definition of L, H, Φ_n, we have Lemma 5.

Lemma 5. If the marking evolution equation of the colored Petri net is $x(k+1) = Lu(k)x(k)$, then

$$A = LH\Phi_n = \sum_{j=1}^{r} \mathrm{Blk}_j(L),$$

where H is the control quantity enable matrix of the colored Petri net $\langle \mathrm{CPN}, M_0\rangle$.

In the general case, the matrix $A = LH\Phi_n$ is not a Boolean matrix; therefore it is not convenient to judge the stability of the colored Petri net $\langle \mathrm{CPN}, M_0\rangle$ by using $x(k+1) = LH\Phi_n x(k)$. We, therefore, consider $x(k+1) = LH\Phi_n x(k)$ in the framework of Boolean algebra. The equation could be expressed as

$$x(k+1) = \tilde{A} \ltimes_{\mathcal{B}} x(k),$$

where $\tilde{A} := L \ltimes_{\mathcal{B}} H \ltimes_{\mathcal{B}} \Phi_n = \mathrm{Blk}_1(L) +_{\mathcal{B}} \mathrm{Blk}_2(L) +_{\mathcal{B}} \cdots +_{\mathcal{B}} \mathrm{Blk}_r(L)$, and $x(k) = (x^1(k), x^2(k), \ldots, x^n(k))^{\mathrm{T}}$ is the vector form of the marking of step k.

$x^j(k) = 1$ if and only if the initial marking can reach $M_{j-1} = \delta_n^j$ under the input sequence of k steps; otherwise $x^j(t) = 0$. In particular, $x(0) = (x^1(0), x^2(0), \ldots, x^n(0))^{\mathrm{T}}$ is the initial marking of $\langle \mathrm{CPN}, M_0\rangle$.

Theorem 1. Assume the marking evolution equation of $\langle \mathrm{CPN}, M_0\rangle$ is $x(k+1) = \tilde{A} \ltimes_{\mathcal{B}} x(k)$, and $M_{i-1} = \delta_s^i \in R(\mathrm{CPN}, M_0)$. Then the reachable set of k steps from M_{i-1} is

$$R_k(\mathrm{CPN}, M_{i-1}) = \Theta(\mathrm{Col}_i(\tilde{A}^k)),$$

where $\tilde{A}^k$ is the k times Boolean power of $\tilde{A}$.

Proof. As $x(k+1) = \tilde{A} \ltimes_{\mathcal{B}} x(k)$, $(\tilde{A})_{(j,i)} = 1$ iff $M_{j-1} = \delta_s^j$ is reachable from $M_{i-1} = \delta_s^i$. That is there exists a control quantity t_{j_1} such that $M_{i-1}[t_{j_1} > M_{j-1}$. Thus, in the framework of Boolean algebra $(\tilde{A}^k)_{j,i} = 1$, if there exists a control quantity sequence $\delta = t_{j_1} t_{j_2} \cdots t_{j_k} \in T^*$ of length k, such that $M_{i-1}[\sigma > M_{j-1}$, Theorem 1 holds.

Theorem 2. Assume the marking evolution equation of $\langle \mathrm{CPN}, M_0\rangle$ is $x(k+1) = \tilde{A} \ltimes_{\mathcal{B}} x(k)$, and $M_{p-1} := \delta_s^p \in R(\mathrm{CPN}, M_0)$ is a equilibrium point of $\langle \mathrm{CPN}, M_0\rangle$.

(1) The marking $M_{i-1} = \delta_s^i$ is stable at M_{p-1} iff

$$\delta_s^p \in \Theta(\mathrm{Col}_i(\tilde{A}^k)), \quad \forall k \geqslant T_p(M_{i-1}),$$

where $T_p(M_{i-1})$ is the transient period of the marking M_{i-1};

(2) The colored Petri net $\langle \mathrm{CPN}, M_0\rangle$ is stable at M_{p-1} iff

$$\delta_s^p \in \Theta(\mathrm{Col}(\tilde{A}^k)), \quad \forall k \geqslant T_p,$$

where T_p is the transient period of $\langle \mathrm{CPN}, M_0\rangle$.

Proof. Statement (2) can be inferred from (1); therefore, we prove (1) holds.

If the marking $M_{i-1} = \delta_s^i$ is stable at the equilibrium point M_{p-1}, now that $X(M, \delta, k) = \{M_p\}$, $\forall k \geqslant T(M)$, then there exists $T_p(M_{i-1}) \in \mathcal{N}^+$ such that $X(M_{i-1}, \delta, k) = \{M_{p-1}\}$, $\forall k \geqslant T_p(M_{i-1})$. From the above Theorem 1, the reachable set from the marking $M_{i-1} = \delta_s^i$ under greater than or equal to $T_p(M_{i-1})$ steps is $\Theta(\mathrm{Col}_i(\tilde{A}^k)) = \{\delta_s^p\}$, so that (1) holds, which implies that all the state trajectories converge to M_{p-1} under finite steps starting from $M_{i-1} = \delta_s^i$. So the marking $M_{i-1} = \delta_s^i$ is stable at M_{p-1}.

Example 2. Consider the above colored Petri net, and judge the stability at the equilibrium point $M_7 = \delta_{11}^7$.

$$x(k+1) = \tilde{A} \ltimes_{\mathcal{B}} x(k),$$

where

$$\tilde{A} = \begin{bmatrix} 0&0&1&1&1&0&0&0&0&0&0 \\ 1&1&0&0&0&1&1&0&0&0&0 \\ 1&0&0&0&0&0&0&1&1&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&1&0&0 \\ 0&1&1&0&0&1&0&0&0&1&0 \\ 0&1&0&0&1&0&1&0&0&1&0 \\ 0&0&1&0&0&0&0&0&0&0&1 \\ 0&0&1&0&1&0&0&0&0&0&1 \\ 0&0&0&0&0&1&1&0&1&1&0 \\ 0&0&0&0&0&0&0&1&1&0&0 \end{bmatrix}.$$

A positive integer $T_7 \in [1, 11]$ does not exist, such that $\forall k \geqslant T_7$, $\delta_{11}^7 \in \Theta(\mathrm{Col}(\tilde{A}^k))$, so the colored Petri net is not stable at the equilibrium point $M_7 = \delta_{11}^7$.

4 The stabilizability of a colored Petri net

In this section, we study the feedback stabilization of a colored Petri net. First, we study the k-steps pre-reachability set of the colored Petri net and its related properties. By using these properties, we give the condition of stabilization of the colored Petri net and design an algorithm to calculate the feedback controller.

4.1 Pre-reachability set of colored Petri net

Definition 8. Assume $M \in R(\mathrm{CPN}, M_0)$ is a marking of $\langle \mathrm{CPN}, M_0\rangle$. The k-steps pre-reachability set of M, denoted by $PR_k(\mathrm{CPN}, M)$, is defined as $\{M' \in R(\mathrm{CPN}, M_0) \,|\, \exists \delta = t_{j_1} t_{j_2} \cdots t_{j_k} \in T^*$, such that $M'[\delta > M\}$, that is

$$PR_k(\mathrm{CPN}, M) = \{M' \in R(\mathrm{CPN}, M_0) \,|\, \exists \delta = t_{j_1} t_{j_2} \cdots t_{j_k} \in T^*, \text{ such that } M'[\delta > M\}.$$

Proposition 1. Assume $M \in R(\mathrm{CPN}, M_0)$ is a marking of the colored Petri net $\langle \mathrm{CPN}, M_0\rangle$ and $M \in PR_1(\mathrm{CPN}, M)$. Then

$$PR_k(\mathrm{CPN}, M) \subseteq PR_{k+1}(\mathrm{CPN}, M), \quad \forall k \geqslant 1.$$

Proof. If the k-steps pre-reachability set of M is $PR_k(\mathrm{CPN}, M)$, then $\forall M_i \in PR_k(\mathrm{CPN}, M)$, M_i could reach M by k steps, and M could reach M by 0 step. M_i could reach M by $k+1$ steps, that is, $M_i \in PR_{k+1}(\mathrm{CPN}, M)$; therefore $PR_k(\mathrm{CPN}, M) \subseteq PR_{k+1}(\mathrm{CPN}, M)$, $\forall k \geqslant 1$.

Proposition 2. Assume $M \in R(\mathrm{CPN}, M_0)$ is a marking of $\langle \mathrm{CPN}, M_0\rangle$.

(1) If $PR_1(\mathrm{CPN}, M) = \{M\}$, then $PR_k(\mathrm{CPN}, M) = \{M\}$, $\forall k \geqslant 1$;

(2) If there exists $i \in \mathcal{N}^+$ such that $PR_i(\mathrm{CPN}, M) = PR_{i+1}(\mathrm{CPN}, M)$, then $PR_k(\mathrm{CPN}, M) = PR_i(\mathrm{CPN}, M)$, $\forall k \geqslant i$.

Proof. Statement (1) obviously holds. Next we prove that (2) holds.

We only need to prove that (2) holds when $k = i+2$; other cases can be proved in a similar way. On the one hand, because $PR_i(\mathrm{CPN}, M) = PR_{i+1}(\mathrm{CPN}, M)$, $PR_{i+1}(\mathrm{CPN}, M) \subseteq PR_{i+2}(\mathrm{CPN}, M)$, that is

$$PR_i(\mathrm{CPN}, M) \subseteq PR_{i+2}(\mathrm{CPN}, M).$$

On the other hand, if the marking $M' \in PR_{i+2}(\mathrm{CPN}, M)$, then there exists a firing sequence $\delta = t_{j_1} t_{j_2} \cdots t_{j_i} t_{j_{i+1}} t_{j_{i+2}} \in T^*$, such that $M'[\delta > M$. There must exist two markings $M'' \in PR_{i+1}(\mathrm{CPN}, M)$ and $M''' \in PR_i(\mathrm{CPN}, M)$, such that $M''[t_{j_2} > M'''$ and $M'[t_{j_1} > M''$. Because $M'' \in PR_{i+1}(\mathrm{CPN}, M)$

and $PR_i(\mathrm{CPN}, M) = PR_{i+1}(\mathrm{CPN}, M)$, $M'' \in PR_i(\mathrm{CPN}, M)$ and $M' \in PR_{i+1}(\mathrm{CPN}, M)$. Thus $M' \in PR_i(\mathrm{CPN}, M)$, that is

$$PR_{i+2}(\mathrm{CPN}, M) \subseteq PR_i(\mathrm{CPN}, M).$$

So $PR_i(\mathrm{CPN}, M) = PR_{i+2}(\mathrm{CPN}, M)$ holds.

We can calculate the k-steps pre-reachability set of the colored Petri net by using Proposition 3.

Proposition 3. Assume the marking evolution equation of $\langle \mathrm{CPN}, M_0 \rangle$ is $x(k+1) = \tilde{A} \ltimes_{\mathcal{B}} x(k)$ and $M_{i-1} = \delta_s^i \in R(\mathrm{CPN}, M_0)$ is a marking of $\langle \mathrm{CPN}, M_0 \rangle$. Then

$$PR_k(\mathrm{CPN}, M_{i-1}) = \Theta((\mathrm{Row}_i(\tilde{A})^k)^{\mathrm{T}}),$$

where $s = |R(\mathrm{CPN}, M_0)|$, and $\tilde{A}$ is a Boolean matrix.

Proof. From the marking evolution equation $x(k+1) = \tilde{A} \ltimes_{\mathcal{B}} x(k)$, $(\tilde{A})_{(i,j)} = 1$ iff $M_{i-1} = \delta_s^i$ is reachable from $M_{j-1} = \delta_s^j$, that is, there exists an enable control quantity $t_{j_1} \in T$ such that $M_{j-1}[t_{j_1} > M_{i-1}$; then in the framework of Boolean algebra, $(\tilde{A}^k)_{i,j} = 1$ iff there exists an enable control quantity sequence $\delta = t_{j_1} t_{j_2} \cdots t_{j_k} \in T^*$ of k steps, such that $M_{j-1}[\delta > M_{i-1}$. The theorem holds.

Proposition 4. Assume the marking evolution equation of $\langle \mathrm{CPN}, M_0 \rangle$ is $x(k+1) = Lu(k)x(k)$, and $M_{i-1} = \delta_s^i \in R(\mathrm{CPN}, M_0)$ is a marking of $\langle \mathrm{CPN}, M_0 \rangle$, where $s = |R(\mathrm{CPN}, M_0)|$ and the control quantity marking transfer matrix is $L := \delta_s[\alpha_1, \alpha_1, \ldots, \alpha_{rs}]$. Then

(1) $PR_1(\mathrm{CPN}, M_{i-1}) = \{\delta_s^p | \alpha_{(j-1)s+p} = i, 1 \leqslant j \leqslant r\}$;

(2) $PR_{k+1}(\mathrm{CPN}, M_{i-1}) = \cup\{PR_1(\mathrm{CPN}, M') \,|\, M' \in PR_k(\mathrm{CPN}, M_{i-1})\}$, $k = 1, 2, 3, \ldots$.

Proof. The marking evolution equation of colored Petri net is $x(k+1) = Lu(k)x(k)$. Then $L \ltimes \delta_m^j \ltimes \delta_s^p = \delta_s^{\alpha_{(j-1)s+p}}$. $\delta_s^p \in PR_1(\mathrm{CPN}, M_{r-1})$ iff there exists an enable control quantity $t_j = \delta_m^j \in T$, such that $\delta_s^{\alpha_{(j-1)s+p}} = \delta_s^r$; therefore (1) holds. The status of (2) can be concluded from the definition of a pre-reachability set.

4.2 Marking feedback stabilization

Consider the colored Petri net $\langle \mathrm{CPN}, M_0 \rangle$; the marking evolution equation is $x(k+1) = Lu(k)x(k)$. If $\langle \mathrm{CPN}, M_0 \rangle$ is stable, we want to find an optimal marking feedback controller $u(k) = f(x(k))$ under which the colored Petri net could be stabilized to $M \in R(\mathrm{CPN}, M_0)$, where the controller function f has one to one mapping. By resorting to the semi-tensor product of matrices, the marking feedback controller can be expressed as follows:

$$u(k) = Kx(k).$$

We call the logical matrix $K \in \mathcal{M}_{r\times s}$ the marking feedback matrix and it has the following form:

$$K = \delta_r[k_1, k_2, \ldots, k_s],$$

where $s = |R(\mathrm{CPN}, M_0)|$, $k_j \in \{1, 2, \ldots, r\}$, and $1 \leqslant j \leqslant s$.

With the marking feedback controller acting on the $\langle \mathrm{CPN}, M_0 \rangle$, the dynamic behavior of the colored Petri net could be expressed as

$$x(k+1) = LK\Phi_n x(k).$$

If the marking feedback matrix $K = \delta_r[k_1, k_2, \ldots, k_s]$ satisfies $K \in \Theta(H)$, then the marking feedback controller $u(k) = Kx(k)$ is permissible. Let $F := LK\Phi_n$; then the marking feedback matrix is permissible iff $F = LK\Phi_n$ is a logical matrix.

Definition 9. Assume the marking evolution equation of $\langle \mathrm{CPN}, M_0 \rangle$ is $x(k+1) = Lu(k)x(k)$ and $M \in R(\mathrm{CPN}, M_0)$ is an equilibrium point of $\langle \mathrm{CPN}, M_0 \rangle$. If there exists a marking feedback controller, such that $\langle \mathrm{CPN}, M_0 \rangle$ is stable at M, then the colored Petri net could be stabilized to the equilibrium point M.

We could use Theorem 3 to judge the stabilizability of a colored Petri net.

Theorem 3. Assume the marking evolution equation of $\langle \mathrm{CPN}, M_0\rangle$ is $x(k+1) = Lu(k)x(k)$ and $M_{p-1} := \delta_s^p \in R(\mathrm{CPN}, M_0)$ is an equilibrium point. Then the colored Petri net could be stabilized to the equilibrium point M_{p-1} iff there exists $\tau \in \mathcal{N}^+$ $(1 \leqslant \tau \leqslant s-1)$, such that

$$PR_\tau(\mathrm{CPN}, M_{p-1}) = \Delta_s,$$

where Δ_s is the vector form of the pre-reachability set of $\langle \mathrm{CPN}, M_0\rangle$ and $s = |R(\mathrm{CPN}, M_0)|$.

Proof. The necessity is obviously set up. Next, we prove the sufficiency. Assume $PR_\tau(\mathrm{CPN}, M_{p-1}) = \Delta_s$ is set up, where $1 \leqslant \tau \leqslant s-1$.

$$\begin{aligned}\Delta_s &= PR_\tau(\mathrm{CPN}, M_{p-1})\\ &= [PR_\tau(\mathrm{CPN}, M_{p-1}) \setminus PR_{\tau-1}(\mathrm{CPN}, M_{p-1})] \cup [PR_{\tau-1}(\mathrm{CPN}, M_{p-1})]\\ &= [PR_\tau(\mathrm{CPN}, M_{p-1}) \setminus PR_{\tau-1}(\mathrm{CPN}, M_{p-1})] \cup [PR_{\tau-1}(\mathrm{CPN}, M_{p-1}) \setminus PR_{\tau-2}(\mathrm{CPN}, M_{p-1})]\\ &\quad \cup [PR_{\tau-2}(\mathrm{CPN}, M_{p-1})]\\ &= [PR_\tau(\mathrm{CPN}, M_{p-1}) \setminus PR_{\tau-1}(\mathrm{CPN}, M_{p-1})] \cup [PR_{\tau-1}(\mathrm{CPN}, M_{p-1}) \setminus PR_{\tau-2}(\mathrm{CPN}, M_{p-1})]\\ &\quad \cup [PR_{\tau-2}(\mathrm{CPN}, M_{p-1}) \setminus PR_{\tau-3}(\mathrm{CPN}, M_{p-1})] \cup [PR_{\tau-3}(\mathrm{CPN}, M_{p-1})]\\ &\quad \vdots\\ &= [PR_\tau(\mathrm{CPN}, M_{p-1}) \setminus PR_{\tau-1}(\mathrm{CPN}, M_{p-1})] \cup [PR_{\tau-1}(\mathrm{CPN}, M_{p-1}) \setminus PR_{\tau-2}(\mathrm{CPN}, M_{p-1})]\\ &\quad \cup [PR_{\tau-2}(\mathrm{CPN}, M_{p-1}) \setminus PR_{\tau-3}(\mathrm{CPN}, M_{p-1})] \cup \cdots \cup [PR_1(\mathrm{CPN}, M_{p-1})].\end{aligned}$$

For any marking δ_s^i and $i \in \{1, 2, \ldots, s\}$, there exists a unique $l_i \in [1, \tau]$, such that $\delta_s^i \in [PR_{l_i}(\mathrm{CPN}, M_{p-1}) \setminus PR_{l_i-1}(\mathrm{CPN}, M_{p-1})]$, $PR_0(\mathrm{CPN}, M_{p-1}) = \emptyset$.

When $l_i = 1$, there exists $1 \leqslant k_i \leqslant r$, such that $L\delta_r^{k_i}\delta_s^i = \delta_s^{\alpha(k_i-1)s+i} = \delta_s^p$.

When $2 \leqslant l_i \leqslant \tau$, there exists $1 \leqslant k_i \leqslant r$, such that $L\delta_r^{k_i}\delta_s^i = \delta_s^{\alpha(k_i-1)s+i} \notin [PR_{l_i}(\mathrm{CPN}, M_{p-1}) \setminus PR_{l_i-1}(\mathrm{CPN}, M_{p-1})]$. By repeating this process, the marking δ_s^i could stabilize to the equilibrium point δ_s^p.

Assume the marking feedback controller is $k := \delta_r[k_1, k_2, \ldots, k_s]$ and $M_{i-1} = \delta_s^i \in \Delta_s$.

When $M_{i-1} = \delta_s^i \in PR_1(\mathrm{CPN}, M_{p-1})$, $L(k\delta_s^i)\delta_s^i = L\delta_r^{k_i}\delta_s^i = \delta_s^p$.

When $M_{i-1} = \delta_s^i \in [PR_{l_i}(\mathrm{CPN}, M_{p-1}) \setminus PR_{l_i-1}(\mathrm{CPN}, M_{p-1})]$, $2 \leqslant l_i \leqslant \tau$, $L(K\delta_s^i)\delta_s^i = L\delta_r^{k_i}\delta_s^i = \delta_s^{\alpha(k_i-1)s+i} \notin [PR_{l_i}(\mathrm{CPN}, M_{p-1}) \setminus PR_{l_i-1}(\mathrm{CPN}, M_{p-1})]$. By repeating this process, the marking δ_s^i could be stabilized to the equilibrium point δ_s^p. The marking feedback controller could be calculated by Algorithm 1.

Algorithm 1 Calculating the marking feedback controller of a colored Petri net

Assume the marking evolution equation of $\langle \mathrm{CPN}, M_0\rangle$ is $x(k+1) = Lu(k)x(k)$, and $M_{p-1} = \delta_s^p \in R(\mathrm{CPN}, M_0)$ is an equilibrium point of $\langle \mathrm{CPN}, M_0\rangle$. We could calculate the marking feedback controller $u(k) = Kx(k)$ by the following steps. Under the marking feedback controller, the colored Petri net $\langle \mathrm{CPN}, M_0\rangle$ could be stabilized to the equilibrium point M_{p-1}.

Step 1: Calculate the k-steps pre-reachability set $PR_k(\mathrm{CPN}, M_{p-1})$. Verify whether there exists a positive integer $1 \leqslant \tau \leqslant s-1$, such that $PR_\tau(\mathrm{CPN}, M_{p-1}) = \Delta_s$. If τ does not exist, the algorithm terminates. Otherwise, go to the next step.

Step 2: Express the Δ_s as the union of the pre-reachability set.

Step 3: For every $i \in \{1, 2, \ldots, s\}$, use $\delta_s^i \in PR_{l_i}(\mathrm{CPN}, M_{p-1}) \setminus PR_{l_i-1}(\mathrm{CPN}, M_{p-1})$ to calculate l_i.

Step 4: Calculate k_i, $i = 1, 2, \ldots, s$.

Thus, we can obtain the marking feedback controller $u(k) = Kx(k)$.

Example 3. Consider the stabilizability of a colored Petri net at the equilibrium point M_6.

First, we verify whether the colored Petri net is stable at the equilibrium point $M_6 = \delta_{11}^6$. From the above proposition, we can get the pre-reachability set of $M_6 = \delta_{11}^6$, that is

$$\begin{aligned}&PR_1(\mathrm{CPN}, M_6) = \{M_2, M_3, M_6, M_{10}\},\\ &PR_2(\mathrm{CPN}, M_6) = \{M_1, M_2, M_3, M_6, M_7, M_8, M_9, M_{10}\},\\ &PR_3(\mathrm{CPN}, M_6) = \{M_1, M_2, M_3, M_4, M_5, M_6, M_7, M_8, M_9, M_{10}, M_{11}\},\end{aligned}$$

Table 1 Pre-reachability set of M_6

3	2	1	0
M_4,M_5,M_{11}	M_1,M_7,M_8,M_9	M_2,M_3,M_6,M_{10}	M_6

$$PR_3(\mathrm{CPN}, M_6) = \Delta_{11}.$$

Therefore, the colored Petri net can be stabilized to M_6. The pre-reachability set of M_6 can be seen in Table 1.

$$\begin{aligned}
&PR_1(\mathrm{CPN}, M_6) = \{M_2, M_3, M_6, M_{10}\},\\
&PR_2(\mathrm{CPN}, M_6) \setminus PR_1(\mathrm{CPN}, M_6) = \{M_1, M_7, M_8, M_9\},\\
&PR_3(\mathrm{CPN}, M_6) \setminus PR_2(\mathrm{CPN}, M_6) = \{M_4, M_5, M_{11}\}.
\end{aligned}$$

It can be seen that the pre-reachability set is disjoint.

$L = \delta_{11}[4\ 0\ 1\ 0\ 0\ 2\ 0\ 3\ 5\ 7\ 9\ 5\ 7\ 9\ 0\ 0\ 10\ 0\ 11\ 0\ 0\ 0\ 3\ 6\ 8\ 1\ 9\ 0\ 10\ 0\ 11\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 2\ 0\ 3\ 6\ 8\ 2\ 0\ 6\ 0\ 7\ 0\ 0\ 0\ 10\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 2\ 0\ 0\ 0\ 6\ 7\ 0\ 0\ 10\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]$.

When $LK\delta_{11}^4\delta_{11}^4 = \delta_{11}^1$, $k_4 = \delta_8^3$. When $LK\delta_{11}^4\delta_{11}^4 = \delta_{11}^7$, $LK\delta_{11}^4\delta_{11}^4 = \delta_{11}^8$, and $LK\delta_{11}^4\delta_{11}^4 = \delta_{11}^9$, k_4 does not exist.

When $LK\delta_{11}^5\delta_{11}^5 = \delta_{11}^1$, $k_5 = \delta_8^4$; when $LK\delta_{11}^5\delta_{11}^5 = \delta_{11}^7$, $k_5 = \delta_8^5$; when $LK\delta_{11}^5\delta_{11}^5 = \delta_{11}^9$, $k_5 = \delta_8^3$; and when $LK\delta_{11}^5\delta_{11}^5 = \delta_{11}^8$, k_5 does not exist.

When $LK\delta_{11}^{11}\delta_{11}^{11} = \delta_{11}^1$ and $LK\delta_{11}^{11}\delta_{11}^{11} = \delta_{11}^7$, k_{11} does not exist. When $LK\delta_{11}^{11}\delta_{11}^{11} = \delta_{11}^8$, $k_{11} = \delta_8^4$; and when $LK\delta_{11}^{11}\delta_{11}^{11} = \delta_{11}^9$, $k_{11} = \delta_8^1$.

When $LK\delta_{11}^1\delta_{11}^1 = \delta_{11}^2$, $k_1 = \delta_8^5$; when $LK\delta_{11}^1\delta_{11}^1 = \delta_{11}^3$, $k_1 = \delta_8^3$; and when $LK\delta_{11}^1\delta_{11}^1 = \delta_{11}^6$ and $LK\delta_{11}^1\delta_{11}^1 = \delta_{11}^{10}$, k_{11} does not exist.

When $LK\delta_{11}^7\delta_{11}^7 = \delta_{11}^2$, $k_7 = \delta_8^4$; when $LK\delta_{11}^7\delta_{11}^7 = \delta_{11}^{10}$, $k_7 = \delta_8^3$; and when $LK\delta_{11}^7\delta_{11}^7 = \delta_{11}^3$ and $LK\delta_{11}^7\delta_{11}^7 = \delta_{11}^6$, k_7 does not exist.

When $LK\delta_{11}^8\delta_{11}^8 = \delta_{11}^{10}$, $LK\delta_{11}^8\delta_{11}^8 = \delta_{11}^6$, and $LK\delta_{11}^8\delta_{11}^8 = \delta_{11}^2$, k_8 does not exist. When $LK\delta_{11}^8\delta_{11}^8 = \delta_{11}^3$, $k_8 = \delta_8^1$.

When $LK\delta_{11}^9\delta_{11}^9 = \delta_{11}^6$ and $LK\delta_{11}^9\delta_{11}^9 = \delta_{11}^2$, k_9 does not exist. When $LK\delta_{11}^9\delta_{11}^9 = \delta_{11}^3$, $k_9 = \delta_8^4$; and when $LK\delta_{11}^9\delta_{11}^9 = \delta_{11}^{10}$, $k_9 = \delta_8^5$.

When $LK\delta_{11}^2\delta_{11}^2 = \delta_{11}^6$, $k_2 = \delta_8^3$; when $LK\delta_{11}^3\delta_{11}^3 = \delta_{11}^6$, $k_3 = \delta_8^5$; when $LK\delta_{11}^6\delta_{11}^6 = \delta_{11}^6$, $k_6 = \delta_8^7$; and when $LK\delta_{11}^{10}\delta_{11}^{10} = \delta_{11}^6$, $k_{10} = \delta_8^4$. Therefore,

$$K = \begin{bmatrix}
0&0&0&0&0&0&0&1&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&1&1&0&1&0&0&0&0\\
0&0&0&0&1&0&1&0&1&1&1\\
1&0&1&0&1&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}.$$

The marking feedback controller is

$$\forall u(t) = \tilde{K}x(t),$$

where $\tilde{K} \in \Theta(K)$. For example,

$$\tilde{K} = \begin{bmatrix} 0&0&0&0&0&0&0&1&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&0&1&1&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&1&0 \\ 0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}.$$

Assume the current marking is $M_5 = \delta_{11}^5$.

$$L \ltimes k \ltimes \delta_{11}^5 \ltimes \delta_{11}^5 = L \ltimes \delta_8^3 \ltimes \delta_{11}^5 = \delta_{11}^9, \quad L \ltimes k \ltimes \delta_{11}^9 \ltimes \delta_{11}^9 = L \ltimes \delta_8^4 \ltimes \delta_{11}^9 = \delta_{11}^3,$$

$$L \ltimes k \ltimes \delta_{11}^3 \ltimes \delta_{11}^3 = L \ltimes \delta_8^5 \ltimes \delta_{11}^3 = \delta_{11}^6, \quad L \ltimes k \ltimes \delta_{11}^6 \ltimes \delta_{11}^6 = L \ltimes \delta_8^7 \ltimes \delta_{11}^6 = \delta_{11}^6.$$

Then, via the marking sequence $M_5 \to M_9 \to M_3 \to M_6$, M_5 can reach M_6.

Then, $\tilde{K}$ has the following value:

$$\tilde{K} = \begin{bmatrix} 0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&1&1 \\ 1&0&1&0&1&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}.$$

Assume the current marking is $M_{11} = \delta_{11}^{11}$.

$$L \ltimes k \ltimes \delta_{11}^{11} \ltimes \delta_{11}^{11} = L \ltimes \delta_8^4 \ltimes \delta_{11}^{11} = \delta_{11}^8, \quad L \ltimes k \ltimes \delta_{11}^8 \ltimes \delta_{11}^8 = L \ltimes \delta_8^1 \ltimes \delta_{11}^8 = \delta_{11}^3,$$

$$L \ltimes k \ltimes \delta_{11}^3 \ltimes \delta_{11}^3 = L \ltimes \delta_8^5 \ltimes \delta_{11}^3 = \delta_{11}^6, \quad L \ltimes k \ltimes \delta_{11}^6 \ltimes \delta_{11}^6 = L \ltimes \delta_8^7 \ltimes \delta_{11}^6 = \delta_{11}^6.$$

Then, via the marking sequence $M_{11} \to M_8 \to M_3 \to M_6$, M_{11} could reach M_6. There are 162 marking feedback controllers, and via any one of them, the current marking could be stabilized to the equilibrium point M_6.

5 Conclusion

In this paper, a study on the stability and stabilization problem of colored Petri net is described. The condition of stability and stabilization of the equilibrium point based on the semi-tensor product of matrices is given. An algorithm is presented to calculate the marking feedback controller. Two examples are given to illustrate the feasibility and effectiveness of the proposed method. By using the proposed method, we can judge the stability and stabilization of a colored Petri net by the matrix approach.

The focus of the current study has been the stability of a single marking. In the future, we shall study the stability of a marking set, which is also a meaningful topic.

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