A complete discrimination system for polynomials with complex coefficients and its automatic generation*

LIANG Songxin (梁松新) and ZHANG Jingzhong (张景中)

(Institute for Educational Softwares, Guangzhou Normal University, Guangzhou 510400, China)

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Abstract By establishing a complete discrimination system for polynomials, the problem of complete root classification for polynomials with complex coefficients is utterly solved, furthermore, the algorithm obtained is made into a general program in Maple, which enables the complete discrimination system and complete root classification of a polynomial to be automatically generated by computer, without any human intervention. Besides, by using the automatic generation of root classification, a method to determine the positive definiteness of a polynomial in one or two indeterminates is automatically presented.

Keywords: discriminant sequence, revised sign list, root classification, complete discrimination system, positive definiteness.

It is well known that any polynomial of positive degree n over complexes admits exactly n complex roots. But how are the roots classified? Precisely speaking, given a polynomial of positive degree n over complexes, what are the numbers of the real and imaginary roots and their multiplicities?

Newton found that the root classification of $ax^2 + bx + c$ is completely determined by its discriminant $b^2 - 4ac$. However, a single discriminant cannot take such an important role for polynomials of a higher degree. Can one find a set of explicit expressions in terms of the coefficients of a polynomial (which is called the complete discrimination system of the polynomial), by which we can completely determine the root classification of the polynomial?

Researches on complete discrimination system for polynomials was slow-footed. A complete discrimination system for a quartic polynomial with real symbolic coefficients was not achieved until this century. Arnon used that result to derive the positive definite conditions on polynomial $x^4 + px^2 + qx + r$. The earliest researches on discrimination system for a quintic polynomial with complex coefficients can be found in references [1,2].

The problem of explicit criterion for complete root classification of a quintic polynomial with real coefficients had not been solved up to 1995. Furthermore, the problem becomes more complex rapidly as the degree of polynomial increases. In 1996, Yang et al. proposed a general algorithm for establishing a complete discrimination system for real coefficient polynomials of any degree[3], which settled the fundamental problem of real algebra satisfactorily.

Reference [3] gives the theoretical results for obtaining a complete root classification and the condition that each class should satisfy. Furthermore, it gives the algorithm and efficient program to produce the discrimination system for a given polynomial. However, it needs human logical

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analysis to practically produce a complete root classification and to obtain the conditions that each class should satisfy by the theoretical results and the polynomials in the discrimination system of an given polynomial. For example, the analysis of the 23 cases for the root classification of a polynomial of degree 6 over reals in ref. [3], which lacks the term of degree 5, is artificially accomplished. As the degree of polynomial increases, human analysis will not only be more complicated but also be easily mistaken. In fact, it is not easy to answer accurately how many cases there are for the root classification of a 9-degree polynomial with real symbolic coefficients. Thus, it is a necessary and non-trivial task to establish a general algorithm to determine the complete root classification for any polynomials and the condition each class should satisfy by computer.

The number of cases for root classification of a polynomial increases with the degree of the polynomial exponentially, of which the case of polynomials with complex coefficients is faster. If the numbers of cases for root classification of an n-degree polynomial with real coefficients and complex coefficients are denoted by Rn and Cn respectively, then we can see from the following data that the ratio of Cn to Rn also increases as the degree n increases:

n	2	3	4	5	10	15	20	25	30	45
Rn	3	4	9	12	118	651	3 177	12 584	46 092	1 353 106
Cn	6	12	27	50	888	9 072	69 545	433 054	2 324 844	192 929 402
Cn/Rn	2	3	3	4	8	14	22	34	50	143

Thus the analysis for the case of polynomials with complex coefficients is more complicated than that of the case of polynomials with real coefficients. It is necessary to establish an algorithm of automatic reasoning.

1 Preliminary

In the following, we denote the greatest common divisor of $f(x)$ and $g(x)$ by $\gcd(f,g)$, the square root of -1 by I. As we know, there are two kinds of roots for polynomials over reals: real roots and conjugate imaginary roots; whereas, there are three kinds of roots for a polynomial over complexes: real roots, conjugate imaginary roots and non-conjugate imaginary roots. What is their distribution?

Theorem 1.1. *Let $f(x)=f_1(x)+I*f_2(x)$, where $f_i(x)$ is over reals, $i=1,2$, and $h(x)=\gcd(f_1(x),f_2(x))$. Then the set of all roots of $h(x)$ exactly consists of all real roots and all conjugate imaginary roots of $f(x)$.*

Proof. Let a be a real root of $f(x)$ with multiplicities e. Then we have $(x-a)^e \mid f_1(x)$, $(x-a)^e \mid f_2(x)$ by the fact that $(x-a)^e \mid f(x)$ and that $(x-a)^e$ is a polynomial with real coefficients, and so $(x-a)^e \mid h(x)$. On the other hand, we have $h(x) \mid f(x)$. Thus a is exactly a real root of $h(x)$ with multiplicities e.

In the same way, we can prove that, if $a+b*I$ and $a-b*I$ are a pair of conjugate imaginary roots of $f(x)$ with multiplicities r, then they are exactly a pair of conjugate imaginary roots of $h(x)$ with multiplicities r.

We get immediately from Theorem 1.1 the following corollary.

Corollary 1.1. *Let $g(x)=g_1(x)+I^*g_2(x)$, where $g_i(x)$ is over reals, $i=1,2$. If*

$\gcd(g_1, g_2) = 1$, *then* $g(x)$ *has at most non-conjugate imaginary roots.*

By Theorem 1.1 and Corollary 1.1, we can carry out the root classification of a polynomial $f(x)$ over complexes as follows: find out the g.c.d. $p(x)$ of the real part and imaginary part of $f(x)$, then compute the pseudo quotient $g(x)$ of $f(x)$ divided by $p(x)$, and then carry out the root classification of $p(x)$ and $g(x)$ respectively.

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ and $g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0$ be two polynomials over complexes, and let $\{s_i(f,g)\}_i$ and $\{p_i(f,g,x)\}_i$ be the principal subresultant sequence and the subresultant polynomial sequence of $f(x)$ and $g(x)$, respectively (see ref. [4]). We have:

Theorem 1.2. *If* $a_n \neq 0$ *or* $b_m \neq 0$, *and* $s_0(f,g) = s_1(f,g) = \cdots = s_{k-1}(f,g) = 0$, $s_k(f, g) \neq 0$, *then* $\gcd(f, g, x) = p_k(f, g, x)$.

2 Fundamental concepts and results

From now on, by polynomials we always refer to polynomials with complex coefficients except otherwise stated, and by polynomials with imaginary coefficients we refer to such polynomials as $g(x)$, $g(x) = g_1(x) + I * g_2(x)$, where $g_i(x)$ is over reals, $i = 1,2$, and $\gcd(g_1(x), g_2(x)) = 1$.

Definition 2.1. Let $f(x)$ be a polynomial of degree n. The Sylvester matrix of $f(x)$ and $f'(x)$ is called the discrimination matrix of $f(x)$, and denoted by Discr(f). D_k denotes the determinant of the submatrix of Discr(f), formed by the first $2k$ rows and the first $2k$ columns, for $k = 1, 2, \cdots, n$.

Definition 2.2. The n-tuple $[D_1, D_2, \cdots, D_n]$ is called the discriminant sequence of $f(x)$.

Given a polynomial $f(x) = f_1(x) + I * f_2(x)$, where $f_i(x)$ is over reals, $i = 1,2$, let $p(x)$ be the g.c.d. of $f_1(x)$ and $f_2(x)$, and $g(x)$ be the quotient of $f(x)$ divided by $p(x)$.

Definition 2.3. Let $D = [D_1, D_2, \cdots, D_n]$ be the discriminant sequence of $p(x)$, we call the list $[\text{sign}(D_1), \text{sign}(D_2), \cdots, \text{sign}(D_n)]$ the sign list of D.

For the sign list $[s_1, s_2, \cdots, s_n]$ of D, we construct a new list $[\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n]$ as follows, which is called the revised sign list with respect to $p(x)$ (abbr. to r.s.l. wrt $p(x)$).

If $[s_i, s_{i+1}, \cdots, s_{i+j}]$ is a section of the given list, where $s_i \neq 0$, $s_{i+1} = s_{i+2} = \cdots = s_{i+j-1} = 0$, $s_{i+j} \neq 0$, then we replace the subsection $[s_{i+1}, s_{i+2}, \cdots, s_{i+j-1}]$ by $[-s_i, -s_i, s_i, s_i, -s_i, -s_i, s_i, s_i, \cdots]$, i.e. let $\varepsilon_{i+r} = (-1)^{\left[\frac{r+1}{2}\right]} \cdot s_i$ for $r = 1, 2, \cdots, j-1$. Otherwise, let $\varepsilon_k = s_k$, i.e. there are no changes for other terms.

Definition 2.4. Let $[d_1, d_2, \cdots, d_t]$ be the discriminant sequence of $g(x)$, a polynomial with imaginary coefficients, If k is the maximal subscript such that $d_k \neq 0$, then we call the list $[1, 1, \cdots, 1, 0, 0, \cdots, 0]$ (there are k continuous 1's followed by $t-k$ continuous 0's) the revised sign list with respect to $g(x)$ (abbr. to r.s.l. wrt $g(x)$).

Definition 2.5. We call the subresultant polynomial sequence of $f(x)$ and $f'(x)$ the multiple factor sequence of $f(x)$, and denote it by $\{\Delta_0(f), \Delta_1(f), \cdots, \Delta_{n-1}(f)\}$. We have the following:

Proposition 2.1. *If the number of the* 0's *in r.s.l. wrt* $f(x)$ *is* k, *then* $\gcd(f(x), f'(x)) = \Delta_k(f)$.

Proof. It is sufficient to let $g(x)=f'(x)$ in Theorem 1.2.

Given a polynomial $p(x)$ over reals, if we only want to know the number of the distinct real or imaginary roots, then the following propositions are sufficient (see reference [3]).

Proposition 2.2. *Given a polynomial $p(x)$ over reals, if the number of sign changes and the number of non-vanishing members of the r.s.l. wrt $p(x)$ are v and s respectively, then the number of pairs of the distinct conjugate imaginary roots of $p(x)$ is v, and the number of the distinct real roots is $s-2v$.*

Proposition 2.3. *Let $g(x)$ be an imaginary coefficient polynomial of degree n. If the number of non-vanishing members of the r.s.l. wrt $g(x)$ is r, then the number of the distinct non-conjugate imaginary roots of $g(x)$ is r.*

If we want to know not only the number of the distinct roots of a polynomial $f(x)$, but also the multiplicities of every root, then we need to consider the root classification of the "repeated part" of $f(x)$, i.e. $\gcd(f(x),f'(x))$.

Definition 2.6. For convenience, $\Delta(f)$ denotes $\gcd(f(x),f'(x))$, and let $\Delta^0(f)=f(x)$, $\Delta^j(f)=\Delta(\Delta^{j-1}(f))$, for $j=1, 2, \cdots$. We call $\{\Delta^0(f), \Delta^1(f), \Delta^2(f), \cdots\}$ the Δ-sequence of $f(x)$.

It is obvious that the Δ-sequence of a given polynomial is a finite sequence. Given a polynomial $f(x)=f_1(x)+I*f_2(x)$, let $p(x)$ be the g.c.d. of $f_1(x)$ and $f_2(x)$, and $g(x)$ be the quotient of $f(x)$ divided by $p(x)$. Let U and V be the Δ-sequences of $p(x)$ and $g(x)$ respectively, W the union of U and V. We will see later that the discriminant sequences of all polynomials in W will form the complete discrimination system of $f(x)$.

Proposition 2.4. *Let $f(x)$ be a polynomial over complexes. If $\Delta^j(f)$ has k distinct roots with respective multiplicities $n_1, n_2, \cdots, n_k$, then the "repeated part" $\Delta^{j+1}(f)$ of $\Delta^j(f)$ has at most k distinct roots with respective multiplicities n_1-1, n_2-1, $\cdots$, n_k-1 (if the multiplicity of a root is 0, it means the root does not exist).*

Proposition 2.5. *Let $f(x)$ be a polynomial over complexes. If $\Delta^j(f)$ has k distinct roots with respective multiplicities $n_1, n_2, \cdots, n_k$, and $\Delta^{j-1}(f)$ has m distinct roots, then $m\geqslant k$, and the multiplicities of these m distinct roots are n_1+1, $n_2+1, \cdots$, n_k+1, 1, $\cdots$, 1 (there are $m-k$ continuous 1's) respectively.*

3 Root classification for polynomials with constant coefficients

We use a list to express the classification of roots for a given polynomial in this form of "[[the classification of real roots], [the classification of conjugate imaginary roots], [the classification of non-conjugate imaginary roots]]", and call it a list of root classification of the given polynomial. For example, the list of root classification of $f(x)=(x-1)^3(x+2)^2(x^2+1)^5(x^2+x+2)^2(x-3I+5)^3(x+I)$ is [[3, 2], [5, −5, 2, −2], [3, 1]].

It is obvious that there is only one list of root classification for a polynomial with constant coefficients because its roots are completely determined by its coefficients. Now we present an algorithm to determine the root classification of a polynomial with constant coefficients.

Step 1. For polynomial $f(x)=f_1(x)+I*f_2(x)$ where $f_i(x)$ is over reals, we find out the g.c.d. $p(x)$ of $f_1(x)$ and $f_2(x)$, compute the quotient $g(x)$ of $f(x)$ divided by $p(x)$. For $p(x)$ and $g(x)$ we do the following respectively.

Step 2. Find out the r.s.l. wrt $p(x)$, compute the number of sign changes of it to determine the numbers of the distinct conjugate imaginary roots and real roots of $p(x)$; find out the r. s. l. wrt $g(x)$, compute the number of non-vanishing members of it to determine the number of the distinct non-conjugate imaginary roots of $g(x)$. If the r. s. l. wrt $p(x)$ (wrt $g(x)$ resp.) contains no 0, stop.

Step 3. If the r. s. l. wrt $p(x)$ (wrt $g(x)$ resp.) above contains k 0's, then $\Delta(p) = \Delta_k(p)$ ($\Delta(g) = \Delta_k(g)$ resp.), which can be found by the definition of multiple factor sequence. For $\Delta(p)$ ($\Delta(g)$ resp.) do what we did for $p(x)$ ($g(x)$ resp.) in Step 2.

Step 4. Do in the same way for $\Delta^2(p), \Delta^3(p), \cdots (\Delta^2(g), \Delta^3(g), \cdots$ resp.), until for some j, the r. s. l. wrt $\Delta^j(p)$(wrt $\Delta^j(g)$ resp.) contains no 0.

Step 5. Compute the numbers of the distinct roots of all kinds of $\Delta^j(p)$ ($\Delta^j(g)$ resp.) using the method we did in Step 2. Then, compute the numbers with multiplicities of the distinct roots of all kinds of $\Delta^{j-1}(p)$ ($\Delta^{j-1}(g)$ resp.) using Proposition 2.5, and then for $\Delta^{j-2}(p)$ ($\Delta^{j-2}(g)$ resp.), etc., until we obtain the complete root classification of $p(x)$ ($g(x)$ resp.).

The following is the root classification of polynomial $x^{18} - x^{16} + 2x^{15} - x^{14} - x^5 + x^4 + x^3 - 3x^2 + 3x - 1$ generated by computer.

The discriminant sequence of x^18 - x^16 + 2 * x^15 - x^14 - x^5 + x^4 + x^3 - 3 * x^2 + 3 * x - 1 is:

[1, 1, -1, -1, -1, 0, 0, 0, -1, 1, 1, -1, -1, 1, -1, -1, 0, 0].

The root classification of x^18 - x^16 + 2 * x^15 - x^14 - x^5 + x^4 + x^3 - 3 * x^2 + 3 * x - 1 is: ([[real roots], [conjugate imaginary roots], [non-conjugate imaginary roots]])

[[1, 1], [2, -2, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1], []].

4 All possible root classifications for a polynomial

From now on, we want to discuss the problem of root classification for polynomials with symbolic/literal coefficients. Given a polynomial of degree n with symbolic coefficients, from the theoretical point of view, how many possible root classifications will it have?

4.1 All possible root classifications for a polynomial of degree n over reals

As we know, a polynomial of degree n over reals admits exactly n roots, which can be divided into two types: real roots and conjugate imaginary roots, and conjugate imaginary roots always exist in pairs. Thus all lists of root classifications for a polynomial of degree n over reals can be expressed as: {all lists of root classifications with 0 pair of conjugate imaginary roots} $\cup$ {all lists of root classifications with 1 pair of conjugate imaginary roots} $\cup \cdots \cup$ {all lists of root classifications with $\left[\frac{n}{2}\right]$ pairs of conjugate imaginary roots}.

For $0 \leqslant k \leqslant \left[\frac{n}{2}\right]$, in order to find out all lists of root classifications with k pairs of conjugate imaginary roots for a polynomial of degree n over reals, noticing that it has $n - 2k$ real roots at the moment, we take the following steps:

(S1) Find out all splits A of the natural number $n - 2k$;

(S2) find out all splits B of the natural number k;

(S3) for $\forall b \in B$, we make mapping: $i \mapsto i, -i$, for $\forall i \in b$;

(S4) combine the elements of A with the elements of B by using distribution law.

Based on the algorithm above, we have had a program written in Maple to find out all possible lists of root classifications for a polynomial of degree n over reals.

4.2 All possible root classifications for a polynomial of degree n over complexes

Given an n-degree polynomial $f(x)=f_1(x)+I*f_2(x)$, let $p(x)$ be the g.c.d. of $f_1(x)$ and $f_2(x)$, $g(x)$ be the quotient of $f(x)$ divided by $p(x)$. Let $\deg(g(x),x)=k$. Then all lists of root classifications of $f(x)$ can be expressed as: {all lists of root classifications of $f(x)$ when $k=0$} ∪ {all lists of root classifications of $f(x)$ when $k=1$} ∪⋯∪ {all lists of root classifications of $f(x)$ when $k=n$}.

For $0\leqslant k\leqslant n$, in order to find out all lists of root classifications with k non-conjugate imaginary roots for a polynomial of degree n over complexes, noticing that the degree of $p(x)$ is $n-k$ at the moment, we take the following steps:

(S1) Find out all lists of root classifications A of a polynomial of degree $n-k$ over reals;

(S2) find out all splits B of the natural number k;

(S3) combine the elements of A with the elements of B by using distribution law.

Based on the algorithm above, we have had a program written in Maple to find out all possible lists of root classifications for a polynomial of degree n over complexes.

5 From list of root classification to revised sign list

Given a polynomial $f(x)=f_1(x)+I*f_2(x)$, let $p(x)$ be the g.c.d. of $f_1(x)$ and $f_2(x)$, $g(x)$ be the quotient of $f(x)$ divided by $p(x)$.

For a polynomial with constant coefficients, its roots are completely determined, and so it has only one list of root classification. However, for a polynomial with symbolic coefficients, its root classification is undetermined because its symbolic coefficients are undetermined. Thus it may have more than one list of root classification. So, in this case, it will not work to determine root classification by the polynomial. We must consider this problem from a different angle, that is, to determine the properties of the polynomial by its list of root classification.

Now, suppose $[b_1,b_2,b_3]$ is a list of root classification of polynomial $f(x)$ with symbolic coefficients. Then, what conditions should $p(x)$ and $g(x)$ satisfy? We can express this problem exactly. In order to make the list of root classification of $p(x)$ exactly $[b_1,b_2]$, what conditions should it satisfy for the r. s. l. wrt the polynomials in the Δ-sequence of $p(x)$? And, in order to make the list of root classification of $g(x)$ exactly b_3, what conditions should it satisfy for the r. s. l. wrt the polynomials in the Δ-sequence of $g(x)$? We learn from sec. 3 that it is sufficient to determine the root classification of a polynomial by the r. s. l. wrt the polynomials in its Δ-sequence.

5.1 From classification list of non-conjugate imaginary roots to revised sign list

Suppose $b_3=[n_1,n_2,\cdots,n_k]$ is a list of root classification of $g(x)$, $n_1\leqslant n_2\leqslant\cdots\leqslant n_k$, and let $t=n_1+n_2+\cdots+n_k$. Then $g(x)$ should be a polynomial of degree t, and it has k distinct roots. Thus, by Proposition 2.3, the r. s. l. wrt $g(x)$ should be $srl_0=[1,\cdots,1,0,\cdots,0]$ (there are k continuous 1's followed by $t-k$ continuous 0's). $[g,srl_0]$ means that the r. s. l. wrt $g(x)$ is srl_0.

1) If $t=k$, i.e. the r. s. l. wrt $g(x)$ contains no 0, then there is only one polynomial in the Δ-sequence of $g(x)$, $g(x)$ itself. Thus it is sufficient by $[g,srl_0]$ to determine that the list of root classification of $g(x)$ is b_3;

2) if $k=1$, i.e. the r.s.l. wrt $g(x)$ contains only one 1, then $g(x)$ has only one distinct root, so do the other polynomials in $\triangle$-sequence of $g(x)$. Thus any r.s.l. wrt the polynomials in $\triangle$-sequence of $g(x)$ also contains only one 1;

3) if $t-k=1$, then the "repeated part" $\Delta^1(g)$ is a polynomial of degree 1, and the r.s.l. wrt it is [1].

For these three cases above, we can determine that the list of root classification of $g(x)$ is b_3, without further computation of the revised sign lists of other polynomials.

If $t>k+1$ and $k\neq 1$, then, by Proposition 2.1, the "repeated part" $\Delta^1(g)$ of $g(x)$ is a polynomial of degree $t-k$. By Proposition 2.4, there is a natural number $r\geqslant 1$ such that the list of root classification of $\Delta^1(g)$ is $[n_r-1, n_{r+1}-1, \cdots, n_k-1]$. Repeating what we did above, we can obtain srl_1, the r.s.l. wrt $\Delta^1(g)$.

For $\Delta^2(g), \Delta^3(g), \cdots$, do what we did above, until for some j, the r.s.l. wrt $\Delta^j(g)$ contains no 0, or only one 1, or only one 0. At that time, we obtain a sequence of polynomials and their revised sign lists $[g, srl_0]$, $[\Delta^1(g), srl_1]$, $[\Delta^2(g), srl_2], \cdots, [\Delta^j(g), srl_j]$, and the list of root classification of $g(x)$ determined by it is exactly b_3.

We have had a program based on the algorithm above. The following examples are generated by computer, where $g7=x^7+ax^5+bx^4+cx^3+dx^2+ex+f$ is a polynomial with imaginary coefficients.

```
> rt_rsl1([3,4], x^7 + a * x^5 + b * x^4 + c * x^3 + d * x^2 + e * x + f, x);
   [[g7, [1, 1, 0, 0, 0, 0, 0]],[g75, [1, 1, 0, 0, 0]],[g753, [1, 1, 0]]];
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(Here, $g75=\Delta^1(g7)$, which is of degree 5; $g753=\Delta^1(g75)$, which is of degree 3).

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> rt_rsl1([7], x^7 + a * x^5 + b * x^4 + c * x^3 + d * x^2 + e * x + f, x);
   [[g7,[1, 0, 0, 0, 0, 0, 0]]].
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5.2 From list of root classification of polynomials over reals to revised sign list

Let $lt=[a_1, a_2, \cdots, a_m]$. We denote by $lt[k]$ the k-th element of lt, i.e. $lt[k]=a_k$. For convenience, we introduce three function notations.

1) *sum_w_ch*——input: a list of root classification of a polynomial over reals; output: the degree, the number of distinct roots and the number of pairs of the distinct conjugate imaginary roots of the given polynomial.

2) *getlt*(m, n, k)——generates all possible (revised sign) lists of which the length, the number of non-vanishing members and the number of sign changes are m, n and k, respectively.

3) *ltminus*——decreases the absolute values of all elements in a list of root classification of a polynomial over reals by 1, and then erases all elements of value 0.

Suppose $[b_1, b_2]$ is a list of root classification of $p(x)$, a polynomial over reals, and let $t=sum\_w\_ch([b_1, b_2])$. Then $p(x)$ should be a polynomial of degree $t[1]$, with $t[2]$ distinct roots, with $t[3]$ pairs of distinct conjugate imaginary roots. Thus, by Proposition 2.2, all possible r.s.l. wrt $p(x)$ are $srl_0=getlt(t[1], t[2], t[3])$.

As in the subsection above, we can determine that the list of root classification of $p(x)$ is $[b_1, b_2]$ without further computation of the revised sign lists of other polynomials for the following four cases: (1) $t[1]=t[2]$; (2) $t[2]=1$; (3) $t[1]-t[2]=1$; (4) $t[2]=2$ and $t[3]=1$.

If $[b_1, b_2]$ is not one of the four cases above, then the "repeated part" $\Delta^1(p)$ is a polynomial of degree $t[1]-t[2]$, and by Proposition 2.4, the list of root classification of $\Delta^1(p)$ is

$ltminus([b_1,b_2])$. Repeating what we did above, we can obtain srl_1, all possible r.s.l. wrt $\Delta^1(p)$.

For $\Delta^2(p), \Delta^3(p), \cdots$, do what we did above, until for some j, the r.s.l. wrt $\Delta^j(p)$ contain no 0, or only one 1, or only one 0, or only two non-vanishing members: 1 and -1. At that time, we obtain a sequence of polynomials and their revised sign lists $[p, srl_0]$, $[\Delta^1(p), srl_1]$, $[\Delta^2(p), srl_2], \cdots, [\Delta^j(p), srl_j]$, and the list of root classification of $p(x)$ determined by this sequence is exactly $[b_1, b_2]$.

We present here two examples generated by computer, where $p8 = x^8 + ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + f$.

> *rt_rsl2*([[1,3],[2, −2]], x^8 + a * x^6 + b * x^5 + c * x^4 + d * x^3 + e * x^2 + f, x);

[[*p*8,[1, 1, 1, −1, 0, 0, 0, 0], [1, 1, −1, −1, 0, 0, 0, 0], [1, −1, −1, −1, 0, 0, 0, 0]],[*p*84,[1, 1, −1, 0], [1, −1, −1, 0]]].

> *rt_rsl2*([[2],[1, −1,2, −2]], x^8 + a * x^6 + b * x^5 + c * x^4 + d * x^3 + e * x^2 + f, x);

[[*p*8,[1, −1, 1, 1, 1, 0, 0, 0], [1, 1, −1, 1, 1, 0, 0, 0], [1, −1, −1, 1, 1, 0, 0, 0], [1, 1, 1, −1, 1, 0, 0, 0], [1, 1, −1, −1, 1, 0, 0, 0], [1, −1, −1, −1, 1, 0, 0, 0]],[*p*83,[1, 1, −1], [1, −1, −1]]].

6 Generation of complete discrimination system for polynomials with real symbolic coefficients

Given a natural number n, we present here an algorithm to generate the complete discrimination system and the complete root classification for a polynomial of degree n over reals.

Step 1. Find out all lists of root classifications with 0 non-conjugate imaginary root for a polynomial of degree n (see section 4).

Step 2. For every list of root classification, find out the sequence of r.s.l. wrt it (see section 5).

Step 3. Print out every list of root classification and the sequence of r.s.l. wrt it in order.

Our algorithm is general and efficient. However, due to limited space, we present here only the complete discrimination system and the complete root classification for a polynomial of degree 6 over reals, which is completely and automatically generated by computer. Though the result here and that of ref. [3] are exactly the same, the difference is that the result here is completely and automatically generated by computer while that of ref. [3] is accomplished by human analysis.

> *allflb*(x^6 + p * x^4 + q * x^3 + r * x^2 + s * x + t, x);

(*) *p*6 := x^6 + p * x^4 + q * x^3 + r * x^2 + s * x + t.

The root classifications of *p*6 are:

(1) [[6], [ ], [ ]] ([[real roots], [conjugate imaginary roots], [non-conjugate imaginary roots]]), if we have: [*p*6, [1, 0, 0, 0, 0, 0]] ([poly, its all possible revised sign lists])

(2) [[3,3],[],[]], if we have: [*p*6, [1, 1, 0, 0, 0, 0]],[p64, [1, 1, 0, 0]],[*p*642, [1, 1]]

(3) [[2,4],[],[]], if we have: [*p*6,[1, 1, 0, 0, 0, 0]],[*p*64, [1, 1, 0, 0]],[*p*642, [1, 0]]

(4) [[2, 2, 2], [], []], if we have: [*p*6,[1, 1, 1, 0, 0, 0]],[*p*63, [1, 1, 1]]

(5) [[1, 5], [], []], if we have: [*p*6, [1, 1, 0, 0, 0, 0]],[*p*64, [1, 0, 0, 0]]

(6) [[1, 2, 3], [], []],if we have:[p6, [1, 1, 1, 0, 0, 0]],[p63, [1, 1, 0]]

(7) [[1, 1, 4], [], []],if we have:[p6, [1, 1, 1, 0, 0, 0]],[p63, [1, 0, 0]]

(8) [[1, 1, 2, 2], [], []],if we have:[p6, [1, 1, 1, 1, 0, 0]],[p62, [1, 1]]

(9) [[1, 1, 1, 3], [], []],if we have:[p6, [1, 1, 1, 1, 0, 0]],[p62, [1, 0]]

(10) [[1, 1, 1, 1, 2], [], []],if we have:[p6, [1, 1, 1, 1, 1, 0]]

(11) [[1, 1, 1, 1, 1, 1], [], []],if we have:[p6, [1, 1, 1, 1, 1, 1]]

(12) [[1, 1, 1, 1], [1, −1], []],if we have: [p6, [1, 1, 1, 1, 1, −1], [1, 1, 1, 1, −1, −1], [1, 1, 1, −1, −1, −1], [1, 1, −1, −1, −1, −1], [1, −1, −1, −1, −1, −1]]

(13) [[1, 1, 2], [1, −1], []],if we have: [p6, [1, 1, 1, 1, −1, 0], [1, 1, 1, −1, −1, 0], [1, 1, −1, −1, −1, 0], [1, −1, −1, −1, −1, 0]]

(14) [[1, 3], [1, −1], []],if we have: [p6, [1, 1, 1, −1, 0, 0], [1, 1, −1, −1, 0, 0], [1, −1, −1, −1, 0, 0]],[p62, [1, 0]]

(15) [[2, 2], [1, −1], []],if we have: [p6, [1, 1, 1, −1, 0, 0], [1, 1, −1, −1, 0, 0], [1, −1, −1, −1, 0, 0]],[p62, [1, 1]]

(16) [[4], [1, −1], []],if we have: [p6, [1, 1, −1, 0, 0, 0], [1, −1, −1, 0, 0, 0]],[p63, [1, 0, 0]]

(17) [[1, 1], [2, −2], []],if we have: [p6, [1, 1, 1, −1, 0, 0], [1, 1, −1, −1, 0, 0], [1, −1, −1, −1, 0, 0]],[p62, [1, −1]]

(18) [[1, 1], [1, −1, 1, −1], []],if we have: [p6, [1, −1, 1, 1, 1, 1], [1, 1, −1, 1, 1, 1], [1, −1, −1, 1, 1, 1], [1, 1, 1, −1, 1, 1], [1, 1, −1, −1, 1, 1], [1, −1, −1, −1, 1, 1], [1, 1, 1, 1, −1, 1], [1, 1, 1, −1, −1, 1], [1, 1, −1, −1, −1, 1], [1, −1, −1, −1, −1, 1]]

(19) [[2], [2, −2], []],if we have:[p6, [1, 1, −1, 0, 0, 0], [1, −1, −1, 0, 0, 0]],[p63, [1, 1, −1], [1, −1, −1]]

(20) [[2], [1, −1, 1, −1], []],if we have: [p6, [1, −1, 1, 1, 1, 0], [1, 1, −1, 1, 1, 0], [1, −1, −1, 1, 1, 0], [1, 1, 1, −1, 1, 0], [1, 1, −1, −1, 1, 0], [1, −1, −1, −1, 1, 0]]

(21) [[], [3, −3], []],if we have:[p6, [1, −1, 0, 0, 0, 0]]

(22) [[], [1, −1, 2, −2], []],if we have: [p6, [1, −1, 1, 1, 0, 0], [1, 1, −1, 1, 0, 0], [1, −1, −1, 1, 0, 0]],[p62, [1, −1]]

(23) [[], [1, −1, 1, −1, 1, −1], []],if we have: [p6, [1, −1, 1, 1, 1, −1], [1, 1, −1, 1, 1, −1], [1, −1, −1, 1, 1, −1], [1, 1, 1, −1, 1, −1], [1, 1, −1, −1, 1, −1], [1, −1, −1, −1, 1, −1], [1, −1, 1, 1, −1, −1], [1, 1, −1, 1, −1, −1], [1, −1, −1, 1, −1, −1], [1, −1, 1, −1, −1, −1]]

Where,

(#1) p62 := Q1 + Q2 * x + Q3 * x^2,

(#2) p64 := −2 * p * x^4 − 3 * q * x^3 − 4 * r * x^2 − 5 * s * x − 6 * t,

(#3) p642 := −64 * x^2 * p^3 * r + 27 * x^2 * p^2 * q^2 − 120 * x * p^3 * s + 24 * x * r * p^2 * q + 15 * q * p^2 * s − 192 * t * p^3,

(#4) p6 := x^6 + p * x^4 + q * x^3 + r * x^2 + s * x + t,

(#5) p63 := −2 * s * p^2 − 54 * t * q + (− 4 * r * p^2 + 36 * t * p − 45 * s * q) * x + (30 * s * p − 6 * q * p^2 − 36 * r * q) * x^2 + (24 * r * p − 8 * p^3 − 27 * q^2) * x^3.

(Due to limited space, the discriminant sequences of the above polynomials and the expressions of all *Qi* are omitted).

7 Generation of complete discrimination system for polynomials with complex symbolic coefficients

We present here an algorithm to generate the complete discrimination system and the complete root classification for a polynomial of degree n over complexes.

Let $f(x)=f_1(x)+I*f_2(x)$, where $f_i(x)$ is over reals, $i=1,2$.

Step 1. If $\deg(f_1,x)=\deg(f_2,x)$ and the remainder of $f_1(x)$ divided by $f_2(x)$ is 0, or, one of the $f_i(x)$'s is 0, then it can be changed into the case of polynomials with real symbolic coefficients.

Step 2. Compute the principal subresultant sequence $B=[s_0,s_1,s_2,\cdots,s_m]$ and subresultant polynomial sequence $P=[p_0(x),p_1(x),p_2(x),\cdots,p_m(x)]$ of $f_1(x)$ and $f_2(x)$, where $m\leqslant n$.

Step 3. If $s_0=s_1=\cdots=s_{k-1}=0$ and $s_k\neq 0$, then $\gcd(f_1(x),f_2(x))=p_k(x)$. Compute the quotient $g_{n-k}(x)$ of $f(x)$ divided by $p_k(x)$, which is a polynomial of degree $n-k$.

Step 4. Find out all possible lists of root classifications with $n-k$ non-conjugate imaginary roots for a polynomial of degree n. For every list of root classification, find out the sequence of r.s.l. wrt it (see sections 4, 5).

Step 5. Do Step 3 and Step 4 above repeatedly for $k=0, 1, 2, \cdots, m$, until we obtain all lists of root classifications of $f(x)$.

Our algorithm is common and efficient. However, due to limited space, we present here only the complete discrimination system and the complete root classification for a polynomial of degree 4 over complexes, which is completely and automatically generated by computer.

>*allflb*(x^4+a*x^3+b*x^2+c*x+d+(x^4+p*x^3+q*x^2+r*x+s)*I, x);

(#) f := x^4+a*x^3+b*x^2+c*x+d+I*x^4+I*p*x^3+I*q*x^2+I*r*x+I*s.

The real part of f is: f1 := x^4+a*x^3+b*x^2+c*x+d

The imaginary part of f is: f2 := x^4+p*x^3+q*x^2+r*x+s

The principal subresultant sequence of f1 and f2 is:

s0 := det(M40),

s1: (Due to limited space, the expression of s1 is omitted)

s2 := a*c+a*b*p−r*a−q*a^2−c*p−b*p^2+p*r+p*q*a−b^2+2*b*q−q^2,

s3 := a−p,

s4 := 1.

The subresultant polynomial sequence of f1 and f2 is:

p0 := det(M40),

p1 := Q1+Q2*x,

p2 := Q3+Q4*x+Q5*x^2,

p3 := a*x^3+b*x^2+c*x+d−p*x^3−q*x^2−r*x−s,

p4 := x^4+a*x^3+b*x^2+c*x+d,

(**) When $s0<>0$,
The g.c.d. of $f1$ and $f2$ is: $p0 := 1$.
The pseudo quotient of f divided by $p0$ is:

$g4 := x\hat{}4 + a * x\hat{}3 + b * x\hat{}2 + c * x + d + I * (x\hat{}4 + p * x\hat{}3 + q * x\hat{}2 + r * x + s)$

In this case, the root classifications of f are:

(1) [[ ], [ ], [4]] ([[real roots], [conjugate imaginary roots], [non-conjugate imaginary roots]]) if we have: [$g4$, [1, 0, 0, 0]] ([polynomial, its all possible revised sign lists])

(2) [[ ], [ ], [2, 2]], if we have: [$g4$, [1, 1, 0, 0]], [$g42$, [1, 1]]

(3) [[ ], [ ], [1, 3]], if we have: [$g4$, [1, 1, 0, 0]], [$g42$, [1, 0]]

(4) [[ ], [ ], [1, 1, 2]], if we have: [$g4$, [1, 1, 1, 0]]

(5) [[ ], [ ], [1, 1, 1, 1]], if we have: [$g4$, [1, 1, 1, 1]]

Where,
(#1) $g4 := d + I * s + (c + I * r) * x + (b + I * q) * x\hat{}2 + (I * p + a) * x\hat{}3 + (1 + I) * x\hat{}4$,
(#2) $g42 := Q6 + Q7 * x + Q8 * x\hat{}2$.

(**) When $s0 = 0$ and $s1<>0$,
the g.c.d. of $f1$ and $f2$ is: $p1 := Q1 + Q2 * x$.
The pseudo quotient of f divided by $p1$ is:

$g3 := Q9 + Q10 * x + Q11 * x\hat{}\ 2 + (Q2\hat{}\ 3 + I * Q2\hat{}\ 3) * x\hat{}\ 3$

In this case, the root classifications of f are:

(6) [[1], [ ], [1, 1, 1]], if we have: [$g3$, [1, 1, 1]]

(7) [[1], [ ], [1, 2]], if we have: [$g3$, [1, 1, 0]]

(8) [[1], [ ], [3]], if we have: [$g3$, [1, 0, 0]]

Where,
(#1) $g3 := Q9 + Q10 * x + Q11 * x\hat{}\ 2 + Q2\hat{}\ 3 * x\hat{}\ 3 + I * Q2\hat{}\ 3 * x\hat{}\ 3$.

(**) When [$s0 = 0$, $s1 = 0$] and $s2<>0$
The g.c.d. of $f1$ and $f2$ is: $p2 := Q3 + Q4 * x + Q5 * x\hat{}2$
The pseudo quotient of f divided by $p2$ is:

$g2 := Q5\hat{}\ 2 * x\hat{}\ 2 + x * a * Q5\hat{}\ 2 - x * Q4 * Q5 + Q5\hat{}\ 2 * b - Q5 * Q3 - Q4 * Q5 * a + Q4\hat{}\ 2 + I * (Q5\hat{}\ 2 * x\hat{}\ 2 + x * p * Q5\hat{}\ 2 - x * Q4 * Q5 + Q5\hat{}\ 2 * q - Q5 * Q3 - Q4 * Q5 * p + Q4\hat{}\ 2)$

In this case, the root classifications of f are:

(9) [[2], [ ], [1, 1]], if we have: [$p2$, [1, 0]], [$g2$, [1, 1]]

(10) [[1, 1], [ ], [1, 1]], if we have: [$p2$, [1, 1]], [$g2$, [1, 1]]

(11) [[ ], [1, −1], [1, 1]], if we have: [$p2$, [1, −1]], [$g2$, [1, 1]]

(12) [[2], [ ], [2]], if we have: [$p2$, [1, 0]], [$g2$, [1, 0]]

(13) [[1, 1], [ ], [2]], if we have: [$p2$, [1, 1]], [$g2$, [1, 0]]

(14) [[ ], [1, −1], [2]], if we have: [$p2$, [1, −1]], [$g2$, [1, 0]]

Where,
(#1) $p2 := Q3 + Q4 * x + Q5 * x\hat{}\ 2$,
(#2) $g2 := Q12 + Q13 * x + (Q5\hat{}\ 2 + I * Q5\hat{}\ 2) * x\hat{}2$.

(**) When [$s0 = 0$, $s1 = 0$, $s2 = 0$] and $s3<>0$
The g.c.d. of $f1$ and $f2$ is: $p3 := a * x\hat{}3 + b * x\hat{}2 + c * x + d - p * x\hat{}3 - q * x\hat{}2 - r * x -$

s.

The pseudo quotient of f divided by $p3$ is:

$g1 := x*a - x*p + a^2 - a*p - b + q + I*(x*a - x*p + a*p - p^2 - b + q)$

In this case, the root classifications of f are:

(15) [[3], [], [1]], if we have: [$p3$, [1, 0, 0]]

(16) [[1, 2], [], [1]], if we have: [$p3$, [1, 1, 0]]

(17) [[1, 1, 1], [], [1]], if we have: [$p3$, [1, 1, 1]]

(18) [[1], [1, −1], [1]], if we have: [$p3$, [1, 1, −1], [1, −1, −1]]

Where,

(#1) $p3 := a*x^3 + b*x^2 + c*x + d - p*x^3 - q*x^2 - r*x - s$.

(**) When [$s0=0$, $s1=0$, $s2=0$, $s3=0$] and $s4<>0$

The g.c.d. of $f1$ and $f2$ is: $p4 := x^4 + a*x^3 + b*x^2 + c*x + d$

The pseudo quotient of f divided by $p4$ is: $g0 := 1 + I$

In this case, the root classifications of f are:

(19) [[4], [], []], if we have: [$p4$, [1, 0, 0, 0]]

(20) [[2, 2], [], []], if we have: [$p4$, [1, 1, 0, 0]], [$p42$, [1, 1]]

(21) [[1, 3], [], []], if we have: [$p4$, [1, 1, 0, 0]], [$p42$, [1, 0]]

(22) [[1, 1, 2], [], []], if we have: [$p4$, [1, 1, 1, 0]]

(23) [[1, 1, 1, 1], [], []], if we have: [$p4$, [1, 1, 1, 1]]

(24) [[1, 1], [1, −1], []], if we have: [$p4$, [1, 1, 1, −1], [1, 1, −1, −1], [1, −1, −1, −1]]

(25) [[2], [1, −1], []], if we have: [$p4$, [1, 1, −1, 0]; [1, −1, −1, 0]]

(26) [[], [2, −2], []], if we have: [$p4$, [1, −1, 0, 0]]

(27) [[], [1, −1, 1, −1], []], if we have: [$p4$, [1, −1, 1, 1], [1, 1, −1, 1], [1, −1, −1, 1]]

Where,

(#1) $p4 := x^4 + a*x^3 + b*x^2 + c*x + d$.

(#2) $p42 := -8*b*x^2 + 3*x^2*a^2 - 12*c*x + 2*x*b*a + a*c - 16*d$.

(Due to limited space, the discriminant sequences of the above polynomials and the expressions of all Qi are omitted).

8 Root classification for polynomials with sparse symbolic coefficients

For polynomials with sparse symbolic coefficients, or polynomials with fewer symbolic coefficients, some root classifications do not exist. Which of them do not exist? How can we get rid of the root classifications which do not exist as far as possible?

For a concrete polynomial of degree n, $f(x) = f_1(x) + I*f_2(x)$, let $B = [s_0, s_1, s_2, \cdots, s_m]$ and $P = [p_0(x), p_1(x), p_2(x), \cdots, p_m(x)]$ be the principal subresultant sequence and subresultant polynomial sequence of $f_1(x)$ and $f_2(x)$ respectively, where $m \leqslant n$. What we concern first is: For $0 \leqslant k \leqslant m$, is there a g.c.d. of degree k for $f_1(x)$ and $f_2(x)$?

Let $p(x) = \gcd(f_1(x), f_2(x))$. We denote by $g(x)$ the quotient of $f(x)$ divided by $p(x)$, and W the set of all lists of root classifications with $n-k$ non-conjugate imaginary roots for polynomials of degree n. What we concern secondly is: For $b = [b_1, b_2, b_3] \in W$, is b a list of root classification of $f(x)$? Or equivalently, is $[b_1, b_2]$ a list of root classification of $p(x)$, and b_3 a list of root classification of $g(x)$?

Let us discuss the second problem above first. First of all, we would like to discuss the problem of constantization of discriminant sequences whose elements contain symbols. By constantization of discriminant sequence we mean the process of getting all possible discriminant sequences whose elements are all constants, by ranging over the reals for all symbols in the discriminant sequence.

For the discriminant sequence D of a polynomial $p(x)$ over reals, if there are fewer symbols in D, we can obtain the constantization of D by finding out a sample of D-invariant decomposition by cylindrical algebraic decomposition (see ref. [11]), and then substituting the sample points into D one by one. If there are more symbols in D, we can make value assignment for each element of D. The constantization of discriminant sequence of a polynomial $g(x)$ with imaginary coefficients can be done in a similar way as above.

(#1) By the method presented in sec. 5, find out all possible r. s. l. wrt k-degree polynomials over reals and $(n-k)$-degree polynomials with imaginary coefficients having root classification $[b_1, b_2]$ and b_3 resp., denoted by U_1 and V_1 resp. On the other hand, we compute the discriminant sequence D_p and D_g of the concrete polynomials $p(x)$ and $g(x)$ resp. Constantize D_p and D_g resp., and then find out all r. s. l. wrt these constantizations, denoted by U_2 and V_2 resp. If $[b_1, b_2]$ and b_3 are root classifications of $p(x)$ and $g(x)$ resp., then we should have $U_1 \cap U_2 \neq \emptyset$, $V_1 \cap V_2 \neq \emptyset$.

(#2) If the revised sign lists of U_1 contain r (>1) 0's, and the non-vanishing members of them are neither exactly only one 1, nor exactly only one 1 and one -1, then we compute the multiple factor sequence $\{\Delta_0(p), \Delta_1(p), \cdots, \Delta_{k-1}(p)\}$ of $p(x)$. If $[b_1, b_2]$ is a list of root classification of $p(x)$, then we should have $\deg(\Delta_r(p)) = r$. V_1 can be discussed in a similar way.

For $\Delta_r(p)$ and *ltminus*$([b_1, b_2])$, $\Delta_t(g)$ and *ltminus*(b_3), do what we did above. In the process above, if there is any link (#) that is not satisfied, then we can stop discussion and get conclusion that $b = [b_1, b_2, b_3]$ is not a list of root classification of $f(x)$.

For example, let $p(x) = x^5 + x^4 + a$ be a polynomial over reals, and $b_1 = [2, 3]$, $b_2 = [\,]$. Is $[b_1, b_2]$ a list of root classification of $p(x)$?

The set U_1 of all possible r. s. l. wrt a 5-degree polynomial over reals having root classification $[b_1, b_2]$ is $\{[1, 1, 0, 0, 0]\}$; on the other hand, the discriminant sequence of $p(x)$ is $D = [1, 1, 0, -a^2, 256a^3 + 3\,125a^4]$, and $C = \{[1, 1, 0, 0, 0], [1, 1, 0, -1, 1], [1, 1, 0, -1, -1], [1, 1, 0, -1, 0]\}$ is the constantization of D. Thus, the set of all possible r. s. l. wrt $p(x)$ is $U_2 = \{[1, 1, -1, -1, 0], [1, 1, -1, -1, 1], [1, 1, 0, 0, 0], [1, 1, -1, -1, -1]\}$.

Since $U_1 \cap U_2 = \{[1, 1, 0, 0, 0]\} \neq \emptyset$, the first link is satisfied. But as the r. s. l. of U_1 contains 3 0's, we need further to discuss the "repeated part" $\Delta_3(p) = 4x^3 - 25a$. It is indeed a polynomial of degree 3. At this moment, the list of root classification of $\Delta_3(p)$ should be *ltminus*$([b_1, b_2]) = [[1,2], [\,]]$.

It is obvious that the set U_1 of all possible r. s. l. wrt the list of root classification $[[1,2], [\,]]$ is $U_1 = \{[1, 1, 0]\}$; on the other hand, the discriminant sequence of $\Delta_3(p)$ is $D = [1, 0, -a^2]$, and $C = \{[1, 0, 0], [1, 0, -1]\}$ is the constantization of D. Thus, the set of all possible r.s.l. wrt $\Delta_3(p)$ is $U_2 = \{[1, 0, 0], [1, -1, -1]\}$. Since $U_1 \cap U_2 = \emptyset$, [[1,2], []] is not a list of root classification of $\Delta_3(p)$! Thus, [[2,3], []] is not a list of root classification of $p(x)$!

The following is generated by computer, noticing that there are 12 root classifications for polynomials of degree 5 over reals in all, whereas there are only 4 for $x^5 + x^4 + a$!

>*allflb*(x^5 + x^4 + a, x);

(*) p5 := x^5 + x^4 + a

The root classifications of $p5$ are:

(1) [[1, 4], [], []], if we have: [p5, [1, 1, 0, 0, 0]], [p53, [1, 0, 0]]

(2) [[1, 1, 1], [1, -1], []], if we have: [p5, [1, 1, -1, -1, -1]]

(3) [[1, 2], [1, -1], []], if we have: [p5, [1, 1, -1, -1, 0]]

(4) [[1], [1, -1, 1, -1], []], if we have: [p5, [1, 1, -1, -1, 1]]

Where,

(#1) p53 := 4 * x^3 - 25 * a, and its discriminant sequence is: [1, 0, -a^2],

(#2) p5 := x^5 + x^4 + a, and its discriminant sequence is: [1, 1, 0, -a^2, 256 * a^3 + 3 125 * a^4].

We must point out that, for discriminant sequence with more symbols, when we make value assignment for its elements one by one, the process of constantization may include some cases which do not exist. Such deviation may result in that, though some root classifications do not exist, we cannot detect them, and so we cannot erase them. However, the constantization of discriminant sequence is a difficult problem itself. Furthermore, there is something we can ensure that we will not erase a root classification by mistake if it exists.

Now it is time to discuss the first problem above. For $0 \leqslant k \leqslant m$, is there a g.c.d. of degree k for $f_1(x)$ and $f_2(x)$? For convenience, we let the degree of polynomial 0 be -1.

If the degree of $p_k(x)$ in P is smaller than k, then we can declare that there does not exist a g.c.d. of degree k for $f_1(x)$ and $f_2(x)$. In fact, by the definition of subresultant polynomial sequence, if the degree of $p_k(x)$ is smaller than k, then $s_k = 0$. So, "$s_0 = 0, s_1 = 0, \cdots, s_{k-1} = 0$ and $s_k \neq 0$" does not hold.

9 Positive definiteness of polynomials

By using the automatic generation of root classification above, we present here an algorithm to determine the property of a polynomial in one or two indeterminates automatically.

It is simple to determine the positive definiteness (positive semi-definiteness) of a polynomial in one indeterminate. As we know, it is positively definite, if and only if its leading coefficient is greater than 0 and it has no real roots. It is positively semi-definite, if and only if its leading coefficient is greater than 0 and it has no real roots or its every real root has an even multiplicity. Thus, the key is to obtain the root classification of the polynomial.

We denote by $def1(f, x)$ the function for implementing the algorithm above. The return value of $def1(f, x)$ is -1, $-\frac{1}{2}$, 0, $\frac{1}{2}$, 1 and 2 respectively when $f(x)$ is negatively definite, negatively semi-definite, indefinite, positively semi-definite, positively definite and 0.

The determination of property for a polynomial in two indeterminates can be changed into the determination of property for a polynomial in one indeterminate. It is obvious that $f(x, y)$ is positively definite (positively semi-definite), if and only if $f(\alpha, y)$ is positively definite (positively semi-definite) for any real number α.

Swapping x for y if necessary, we can suppose $\deg(f(x,y),x) > \deg(f(x,y),y)$. Compute the discriminant sequence D of $f(x,y)$ with respect to y, $D = [d_1(x), d_2(x), \cdots, d_m(x)]$, denoted by $d(x)$ the product of those $d_k(x)$'s which are not 0. After eliminating the multiple factors and positive definite factors from $d(x)$, we denote it again by $d(x)$.

Without loss of generality, we suppose that $d(x)$ has real roots. Separate all real roots of $d(x)$ by using intervals $[a_1, b_1]$, $[a_2, b_2]$, $\cdots$, $[a_n, b_n]$ with rational ends, where $b_k < a_{k+1}$, $1 \leqslant k \leqslant n-1$. Let $A = \left\{a_1 - 1, \frac{b_1 + a_2}{2}, \frac{b_2 + a_3}{2}, \cdots, \frac{b_{n-1} + a_n}{2}, b_n + 1\right\}$.

We express the real roots of $d(x)$ as algebraic numbers. Let B be the set of all real roots of $d(x)$, $B = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$, where $\alpha_i < \alpha_{i+1}$, and $C = A \cup B = \{\xi_1, \xi_2, \cdots, \xi_r\}$. Then $F = \{(-\infty, \alpha_1), \{\alpha_1\}, (\alpha_1, \alpha_2), \{\alpha_2\}, \cdots, (\alpha_{n-1}, \alpha_n), \{\alpha_n\}, (\alpha_n, +\infty)\}$ is a D-invariant decomposition of real line, and C a sample of F. Furthermore, we have:

Proposition 9.1. *$f(x,y)$ is positively definite (positively semi-definite), if and only if $f(\xi_k, y)$ is positively definite (positively semi-definite) for any $\xi_k \in C$.*

From Proposition 9.1 and the continuity of polynomial functions, we obtain an algorithm to determine the positive definiteness (positive semi-definiteness) of a polynomial $f(x,y)$ in two indeterminates.

Step 1. Compute the discriminant sequence D of $f(x,y)$ with respect to y, $D = [d_1(x), d_2(x), \cdots, d_m(x)]$.

Step 2. Find out a sample $C = \{\xi_1, \xi_2, \cdots, \xi_r\}$ of a D-invariant decomposition.

Step 3. Let *tt* be a Boolean variable with initial value *false*, and S a stack. Compute $def1(f(\xi_k, y), y) = e_k$ for every $\xi_k \in C$, where $1 \leqslant k \leqslant r$. If $e_k = 0$, then stop, and $f(x,y)$ is indefinite; if $e_k = 2$, then assign *tt* the value *true*; otherwise, push e_k into the stack S.

Step 4. Let $P = \{e_k \in S \mid e_k > 0\}$, $N = \{e_k \in S \mid e_k < 0\}$. When P is empty, if *tt* = *true* then returns $-\frac{1}{2}$, otherwise returns the maximum of N; when N is empty, if *tt* = *true* then returns $\frac{1}{2}$, otherwise returns the minimum of P; when neither P nor N is empty, returns 0.

Step 5. Determine the property of $f(x,y)$ by return values. If the return value is -1, $-\frac{1}{2}$, 0, $\frac{1}{2}$ and 1 then $f(x)$ is negatively definite, negatively semi-definite, indefinite, positively semi-definite and positively definite respectively.

In the algorithm above, *tt* = *true* means $f(x,y)$ can be expressed as $(x - \xi_j)^k f_1(x,y)$, where k is a natural number. Thus, *tt* = *true* only means $f(x,y)$ has zeros, the determination of property of $f(x,y)$ needs other $f(\xi_k, y)$'s. We have made the algorithm above into a general program in Maple. It is efficient. The following examples were done on a Pentium586/133 computer with 16 MB RAM.

Example. (1) $x^8 + x^4y^4 - x^2y^2 - x^4 + y^4 + 1$ (positively definite, with CPU time 4 s);

(2) $x^6y^6 - x^4y^3 + x^3y^2 - x^4 - 3y^2 + 1$ (indefinite, 75 s);

(3) $x^6 - x^4y^2 - x^2y^4 + y^6 - x^4 + 3x^2y^2 - y^4 - x^2 - y^2 + 1$ (positively semi-definite, 8 s).

(4) $f(x,y) = x^6y^6 + 6x^6y^5 - 6x^5y^6 + 15x^6y^4 - 36x^5y^5 + 15x^4y^6 + 20x^6y^3 - 90x^5y^4 +$

$90x^4y^5 - 20x^3y^6 + 15x^6y^2 - 120x^5y^3 + 225x^4y^4 - 120x^3y^5 + 15x^2y^6 + 6x^6y - 90x^5y^2 + 300x^4y^3 - 300x^3y^4 + 90x^2y^5 - 6xy^6 + x^6 - 36x^5y + 225x^4y^2 - 400x^3y^3 + 225x^2y^4 - 36xy^5 + y^6 - 6x^5 + 90x^4y - 300x^3y^2 + 300x^2y^3 - 90xy^4 + 6y^5 + 15x^4 - 120x^3y + 225x^2y^2 - 120xy^3 + 15y^4 - 20x^3 + 90x^2y - 90xy^2 + 20y^3 + 16x^2 - 36xy + 16y^2 - 6x + 6y + 1$ (positively definite, 26 s).

Example (3) was also studied in refs. [3,5,6]. Example (4) was studied by other authors[3] in different manners.

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