ON NORMAL CRITERION OF MEROMORPHIC FUNCTIONS

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Abstract

In this paper, the Bloch principle is discussed and a normal criterion is asserted. Let \( \mathcal{F} \) be a family of meromorphic functions on a domain \( D \), \( a \neq 0, \infty; b \neq 0, n \geq 4 \). If for any \( f \in \mathcal{F} \) there exists

\[ f' = af^nb, \]

then \( \mathcal{F} \) is normal in \( D \).

Keywords: meromorphic function, normal family, normal criterion.

I. Introduction

Bloch conjectured that there must exist a normal criterion for every Picard–Liouville theorem.

This principle has played an important role in the theory of distribution value although it is not strict.

Corresponding to this principle the following theorem was established by Zalcman[1].

Theorem A. Let \( P \) be a property of meromorphic functions, satisfying

1) if \( \langle f, D \rangle \in P \), \( D \subset D' \), then \( \langle f, D' \rangle \in P \);

2) if \( \langle f, D \rangle \in P \), \( \phi(z) = az + b \), then \( \langle f, \phi^{-1}(D) \rangle \in P \);

3) if \( \langle f_n, D_n \rangle \in P \) (\( n = 1, 2, \cdots \)), \( D_1 \subset D_2 \subset \cdots \), \( D = \bigcup D_n \) and \( f_n \) uniformly converges to \( f \) on any compact subset of \( D \), then \( \langle f, D \rangle \in P \). Moreover, \( \langle f, D \rangle \in P \) if \( f \) is a constant. Thus for any family \( \mathcal{F} \) of meromorphic functions on a domain \( D \), if for every \( f \in \mathcal{F} \), \( \langle f, D \rangle \in P \), then \( \mathcal{F} \) is normal in \( D \).

In this paper we shall establish

Theorem 1. Let \( \mathcal{F} \) be a family of meromorphic functions on the unit disk that is not a normal family. Then for any real number \( k \) \((-1 < k < 1)\), there exist

1) a real number \( r \), \( 0 < r < 1 \);

2) complex points \( z_n \), \( |z_n| < r \);
iii) functions $f_k \in \mathcal{F}$;

iv) positive numbers $\rho_k$, $\lim \rho_k = 0$,

such that $\frac{f_k(z_0 + \rho_k \xi)}{\rho_k^k}$ uniformly converges to a meromorphic function $g(\xi)$ on any compact subset of $C$, where $g(\xi)$ is nonconstant.

The condition "$-1 < k < 1$" is necessary.

**Theorem 2.** Let $P$ fit a property of meromorphic functions, satisfying

i) if $\langle f, D \rangle \in P$, $D' \subset D$, then $\langle f, D' \rangle \in P$;

ii) for some real number $k (-1 < k < 1)$, if $\langle f, D \rangle \in P$ and $\phi(z) = az + b$,

then $\left( \frac{f \phi}{\phi} \right) \in P$;

iii) if $\langle f, D_n \rangle \in P$ $(n = 1, 2, \cdots)$, $D_1 \subset D_2 \subset \cdots$, $C = \bigcup D_n$ and $f_n$ uniformly converges to $f$ on any compact subset of $C$, then $\langle f, C \rangle \in P$. Moreover, $\langle f, C \rangle \in P$ iff $f$ is a constant. Thus for any family $\mathcal{F}$ of meromorphic functions on any domain $D$, if for every $f \in \mathcal{F}$, there exists $\langle f, D \rangle \in P$, then $\mathcal{F}$ is normal in $D$.

The case $k = 0$ belongs to Zalcman\cite{1}.

The author has proved the case $0 \leq k < 1$ \cite{2}.

Hayman\cite{3} has proved

**Theorem B.** Let $f(x)$ be a meromorphic function on the complex plane, $n \geq 5$, $a \approx 0$, $\infty$ and $b \approx \infty$. If

$$f' - af \approx b,$$

then $f$ is a constant.

According to the Bloch principle, Hayman\cite{4} posed a conjecture: Let $\mathcal{F}$ be a family of meromorphic functions on a domain $D$, $n \geq 5$, $a \approx 0$, $\infty$; $b \approx \infty$. If for any $f \in \mathcal{F}$, there holds

$$f' - af \approx b,$$

then $\mathcal{F}$ is normal in $D$.

Li Xian-jing\cite{5} has confirmed this conjecture.

By some counter examples, Mues\cite{6} disproved Theorem B when $n = 3, 4$. But we have

**Theorem 3.** Let $\mathcal{F}$ be a family of meromorphic functions on domain $D$, $n \geq 4$, $a \approx 0$, $\infty$; $b \approx \infty$. If for any $f \in \mathcal{F}$, there exists

$$f' - af \approx b,$$

then $\mathcal{F}$ is normal in $D$.

**Theorem 4.** Let $\mathcal{F}$ be a family of meromorphic functions on a domain $D$, $n \geq 3$, $a \approx 0$, $\infty$; $b \approx \infty$. If for any $f \in \mathcal{F}$, it follows the multiplicity of zeros of $f \geq 2$, and
then $\mathcal{F}$ is normal in $D$.

II. Lemma

Lemma 1\textsuperscript{(3).} If the family $\mathcal{F}$ of meromorphic functions is not normal in the unit disk $\Delta$, then for every real number $k$ ($0 \leq k < 1$), there exist

i) a real number $r$, $0 < r < 1$;

ii) complex points $z_n$, $|z_n| < r$;

iii) functions $f_n, f \in \mathcal{F}$;

iv) positive numbers $\rho_n$, $\lim_{n \to \infty} \rho_n = 0$, $\lim_{n \to \infty} \frac{r - |z_n|}{\rho_n} = \infty$,

such that $f_n(z_n + \rho_n \xi)$ uniformly converges to a meromorphic function $g(\xi)$ on any compact subset of $C$, where $g(\xi)$ is nonconstant.

III. Proof of the Theorem

Without loss of generality, we may assume that $D = \Delta = \{z; |z| < 1\}$.

Proof of Theorem 1. By Lemma 1, it suffices to show this theorem clearly in the case $-1 < k < 0$. Put

$$k_1 = -k; \mathcal{F}_1 = \{f; \frac{1}{f} \in \mathcal{F}\}.$$  

Then $0 < k_1 < 1$, $\mathcal{F}_1$ is not normal in $\Delta$. By Lemma 1, there exist

i) a real number $r$, $0 < r < 1$;

ii) complex points $z_n$, $|z_n| < r$;

iii) functions $f_n, f \in \mathcal{F}$;

iv) positive numbers $\rho_n$, $\lim_{n \to \infty} \rho_n = 0$, $\lim_{n \to \infty} \frac{r - |z_n|}{\rho_n} = \infty$,

such that $f_n(z_n + \rho_n \xi)$ uniformly converges to a meromorphic function $g(\xi)$ on any compact subset of $C$, where $g(\xi)$ is nonconstant. Put $F_n = \frac{1}{f_n} \in \mathcal{F}$. Then $F_n(z_n + \rho_n \xi)$ uniformly converges to $\frac{1}{g(\xi)}$ on any compact subset of $C$. Clearly, $\frac{1}{g(\xi)}$ is nonconstant.

Proof of Theorem 2. If it is false, $\mathcal{F}$ is not normal. By Theorem 1, $g_n(\xi) = f_n(z_n + \rho_n \xi)$ uniformly converges to a meromorphic function $g(\xi)$ on any compact subset of $C$, where $g(\xi)$ is nonconstant. By the property ii), we have
\[ \langle g, D \rangle \in P, \]

where \( D = \{ \xi; \frac{r - |\xi|}{\rho} \} \).

Since \( \lim_{\rho \to \infty} \frac{r - |\xi|}{\rho} = \infty \), we admit

\[ UD = G. \]

By the property iii), we have

\[ \langle g, C \rangle \in P, \]

and \( g \) is a constant. It is a contradiction. The proof is finished.

**Proof of Theorem 3.** If \( \mathcal{F} \) is not normal in \( \Delta \), by Theorem 1 with \( k = \frac{1}{1 - \alpha} \),

there exist

i) a real number \( r \), \( 0 < r < 1 \);

ii) complex points \( z_i \), \( |z_i| < r \);

iii) functions \( f_i, f_j \in \mathcal{F} \);

iv) positive numbers \( \rho_i \), \( \lim_{i \to \infty} \rho_i = 0 \), \( \lim_{i \to \infty} \frac{r - |z_i|}{\rho_i} = \infty \),

such that \( g_j(\xi) = \rho_j^{-1} \frac{1}{r} f_j(z_i + \rho_i \xi) \) uniformly converges to a meromorphic function

\( g(\xi) \) on any compact subset of \( C \), where \( g(\xi) \) is nonconstant.

On the other hand,

i) if

\[ g'(\xi) = ag''(\xi) \equiv 0, \]

it follows from \( n > 2 \) and \( a \equiv 0 \) that

\[ \frac{g'(\xi)}{g''(\xi)} \equiv a. \]

Letting \( G(\xi) = \frac{1}{g(\xi)} \), we have

\[ G''(\xi) G'(\xi) \equiv -a. \]

Since \( n - 2 \geq 2 \), \( G(\xi) \) is a constant by a result of Mues\(^{46}\).

ii) If there exists \( \xi_0 \) such that

\[ g'(\xi_0) = ag''(\xi_0) = 0, \]

we have \( g(\xi) \approx \infty \). Thus there exists positive number \( \delta \) such that \( g_1(\xi) \) and \( g_2(\xi) \)

are analytic in \( D_{2\delta} = \{ \xi; |\xi - \xi_0| < 2\delta \} \) when \( n \) is large enough. Then \( g_1(\xi) \)

uniformly converges to \( g'(\xi) \) on \( D_{\delta} = \{ \xi; |\xi - \xi_0| < \delta \} \). Thus \( g_1(\xi) = ag_1(\xi) \)

uniformly converges to \( g'(\xi) = ag'(\xi) \). Since
\[ g'(\xi) - ag^*(\xi) = \rho_i^{\frac{n}{n-1}} f_i(x_i + \rho_i \xi) - a\rho_i^{\frac{n}{n-1}} f_i(x_i + \rho_i \xi) \]
\[ = \rho_i^{\frac{n}{n-1}} (f_i(x_i + \rho_i \xi) - af(x_i + \rho_i \xi)) \approx \rho_i^{\frac{n}{n-1}} b, \]
\[ \lim_{i \to \infty} \rho_i^{\frac{n}{n-1}} b = 0, \]
\[ g'_i(\xi) - ag_i^*(\xi) - \rho_i^{\frac{n}{n-1}} b \approx 0, \]
then \( g'(\xi) - ag^*(\xi) \) is identity to zero by the Hurwitz theorem, thus having
\[ g'(\xi) - ag^*(\xi) = 0, \quad \xi \in C. \]
It implies
\[ \frac{1}{1 - n} \cdot \frac{1}{g^{^{(n-1)*}}(\xi)} = a^2 \pm c, \]
where \( c \) is a constant. Then
\[ g(\xi) = \frac{1}{\sqrt[1-n]{(1 - n)(a^2 \pm c)}}. \]
Because of \( n > 2 \) and \( a \neq 0 \), it contradicts that \( g(\xi) \) is a meromorphic function.

Combining i) with ii), we assert the theorem.

Proof of Theorem 4. If \( \mathcal{F} \) is not normal in \( \Delta \), by Theorem 1 with \( k = \frac{1}{1 - n} \),
there exist
i) a real number \( r, \ 0 < r < 1; \)
ii) complex points \( z_i, |z_i| < r; \)
iii) functions \( f_i, |f_i| \in \mathcal{F}; \)
iv) positive numbers, \( \lim_{a \to 0} \rho_i = 0, \lim_{a \to 0} \frac{r - |z_i|}{\rho_i} = \infty, \)
such that \( g_i(\xi) = \rho_i^{\frac{1}{n-1}} f_i(x_i + \rho_i \xi) \) uniformly converges to a meromorphic function \( g(\xi) \) on any compact subset of \( C \), where \( g(\xi) \) is nonconstant.

Similarly, on the other hand, we have
\[ g'(\xi) - ag^*(\xi) \approx 0. \]
Owing to the multiplicity of zeros of \( g(\xi) \geq 2 \), we have \( g(\xi) \approx 0 \). Put \( G(\xi) = \frac{1}{g(\xi)} \). Thus \( G(\xi) \) is an entire function and
\[ g^{n-2}(\xi)G'(\xi) \approx -a, \]
and then \( G(\xi) \) is a constant by a result of Clunie\(^{21}\). This leads to a contradiction.

IV. APPLICATION

Hayman has proved
Theorem C. Let \( f \) be an entire function on the complex plane, \( a \approx 0, \infty; b \approx \infty, f' - af \approx b \). Then \( f \) is a constant.

We can prove

Theorem D. Let \( \mathcal{F} \) be a family of entire functions on a domain \( D \), \( a \approx 0, \infty; b \approx \infty \). If for every \( f \in \mathcal{F} \) we find

\[
f' - af \approx b,
\]

then \( \mathcal{F} \) is normal in \( D \).

**Proof.** We take \( k = -\frac{1}{2} \) and use Theorem 1 and Theorem C.

Hayman has conjectured the following

**Conjecture 1.** Let \( f \) be a meromorphic function on the complex plane.

\[
f' \approx 1,
\]

Then \( f \) is a constant.

If this conjecture is true, then the following two propositions are true, too.

**Proposition 1.** Let \( \mathcal{F} \) be a family of meromorphic functions on the domain \( D \). If for every \( f \in \mathcal{F} \), we find

\[
f' \approx 1,
\]

then \( \mathcal{F} \) is normal in \( D \).

**Proposition 2.** Let \( \mathcal{F} \) be a family of meromorphic functions on domain \( D \), \( a \approx 0, \infty; b \approx \infty \). If for every \( f \in \mathcal{F} \) we have

\[
f' - af \approx b,
\]

then \( \mathcal{F} \) is normal in \( D \).

V. A COUNTER EXAMPLE

The condition "\(-1 < k < 1\)" of Theorem 1 is necessary.

Assume that \( k \geq 1 \). If the theorem is true for \( \mathcal{F} = \{nz\} \), then \( g(z) = f(z + \rho z) \rho \)

uniformly converges to an entire function \( g(z) \) on any compact subset of \( C \), where \( g(z) \) is nonconstant. As

\[
\frac{f_i(z + \rho_i \xi)}{\rho_i^{\frac{1}{k}}} = \frac{n_i}{\rho_i^{\frac{1}{k}}} (z_i + \rho_i \xi),
\]

for a fixed \( \xi \) we have \( \lim_{i \to \infty} |z_i + \rho_i \xi| \leq 1 \). Then it bears

\[
\lim_{i \to \infty} \frac{n_i}{\rho_i^{\frac{1}{k}}} = \lim_{i \to \infty} \frac{n_i}{\rho_i^{\frac{1}{k}}} = \infty.
\]

But
\[
\lim_{i \to \infty} \frac{h_i}{\rho_i^k} z_i = g(0) = \infty,
\]
then
\[
g(1) = \lim_{i \to \infty} \left( \frac{h_i}{\rho_i^k} z_i + \frac{n_j}{\rho_i^{k-1}} \right) = \infty,
\]
which contradicts that \( g(\xi) \) is an entire function.

We put \( \mathcal{E} = \{ \frac{1}{n \xi} \} \) for \( \xi \leq -1 \). The proof is the same as above, so is omitted.

REFERENCES


